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# Pareto optimal anonymous mechanisms

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# Pareto optimal anonymous mechanisms

Conan Mukherjee*

**Abstract**

This paper presents a new characterization of "*maxmed*" mechanisms introduced by Sprumont [26]. This paper, in a two agent setting, shows that maxmed mechanisms are the unique Pareto optimal mechanisms among all mechanisms that satisfy anonymity, strategyproofnes, nonbossiness in decision, feasibility and individual rationality.

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## 1 Introduction

Sprumont [26] studies the important problem of identifying Pareto optimal mechanisms for the single object allotment problem with money. He obtains a remarkable partial result by introducing a new class of "*maxmed"* mechanisms that are the only Pareto optimal mechanisms in the class of anonymous, non-envious, feasible and individually rational strategyproof mechanisms. In the present paper, I provide a similar, but independent characterization of maxmed mechanisms that does not use the axiom of no-envy.[1] In particular, I consider the class of mechanisms that satisfy anonymity in welfare, feasibility, individual rationality, non-bossiness in decision, and strategyproofness. I identify the unique Pareto optimal mechanisms in this class as the class of maxmed mechanisms. I use a two agent setting that can be applied to practical situations like: bilateral trading over an indivisible object between a buyer and a seller, allotment of a government license to private buyers, bankruptcy auction of capital assets by lenders etc.

Anonymity is a popular fairness axiom that requires allocations from a mechanism to any agent be independent of the social identity of the agent, and depend only on the bid values received by the planner.[2] Strategyproofness is a popular strategic axiom

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[1]No-envy is a well known fairness notions that imposes strong technical restrictions on the decision and the transfer functions of a mechanism. It requires that at any state of nature, no agent strictly prefer the allocation bundle of another agent than her own allocation bundle from the mechanism.

[2]This axiom has been used by other papers like Ashlagi and Serizawa [2], Hashimoto and Saitoh [14], Basu and Mukherjee [4] etc. in the related literature.

that requires mechanims to induce agents to bid their true valuations in the ensuing message game, while non-bossiness in decision requires that no agent be able to affect the allocation decision of another agent without affecting her own allocation decision.[3] Feasibility requires that the mechanism not be wasteful so that the sum of monetary transfers never exceed zero, while individual rationality requires that the utility obtained from the mechanism by any agent always be non-negative.

Four papers that consider the related problem of welfare maximization without imposing decision efficiency are: de Clippel, Naroditskiy and Greenwald [6], Drexl and Kleiner [7], Long, Mishra and Sharma [15], Shao and Zhu [25].[4] Like this paper, Drexl and Kleiner [7] focuses on strategyproof, individually rational and feasible mechanisms in a two agent setting, and uses a prior distribution to identify the expected aggregate utility maximizing mechanism. Shao and Zhu [25] obtain similar results as Drexl and Kleiner [7] without the use of individual rationality, but with a more restrictive distribution of types. Long, Mishra and Sharma [15], on the other hand, considers a single object allocation setting with more than three agents to study budget balanced and strategyproof stochastic mechanisms that assign higher allotment probabilities to higher valuation agents. They identify an optimal stochastic mechanism that gives greatest allotment probability to the highest valuation agent.[5] de Clippel, Naroditskiy and Greenwald [6] present a feasible, anonymous, strategyproof, individually rational and decision inefficient mechanism that distributes at least eighty percent of the social welfare generated as number of agents goes to infinity.

However, the papers that are closest to ours are Sprumont [26], and Athanasiou [3]. As mentioned earlier, Sprumont [26] characterizes maxmed mechanisms for allocating an object among $n \geq 2$ agents using the no-envy axiom. Athanasiou [3] does not use the no-envy axiom, and presents necessary conditions for Pareto optimality in the same setting. For $n = 2$, Athanasiou [3] shows that maxmed mechanisms are Pareto optimal among all anonymous, strategyproof, feasible and individually rational mechanisms - but does not prove that there are no other Pareto optimal mechanisms. This paper, like Athanasiou [3], eschews no-envy, but uses non-bossiness in decision to completely characterize maxmed mechanisms in the two agent setting.[6] In particular, I use the single

[3] Note that 'non-bossiness in decision' does not impose any restriction on transfers of a mechanism unlike the conventional version of non-bossiness used by Satterthwaite and Sonnenschein [24]. Similar notions of non-bossiness have been used by Svensson [27], Goswami, Mitra and Sen [9], and Mishra and Quadir [17]. Basu and Mukherjee [4] shows that any strategyproof mechanism violating non-bossiness in decision, would also violate the conventional Satterthwaite and Sonnenschein [24] non-bossiness.

[4] Some other papers investigating the Pareto frontier of mechanisms to allocate indivisible object with money are: Apt, Conitzer, Guo and Markakis [1], Guo and Conitzer [11], Guo and Conitzer [12], Ohseto [23], Moulin [18], Moulin [19]. However, all these papers consider decision efficient mechanisms and hence, limit their study to the class of VCG mechanisms (Vickrey [29], Clarke [5], Groves [10]).

[5] For the two agent case, the optimal mechanism described by Long, Mishra and Sharma [15] does not need use of any transfers.

[6] I show in section 4 that no-envy and non-bossiness in decision are logically independent axioms. In terms of mathematical restrictions, unlike no-envy, non-bossiness in decision only restricts the allocation

object characterization of mechanisms using a reserve price object allocation rule, which was presented in Basu and Mukherjee [4]; and use the additional axioms of feasibility, individual rationality, and Pareto optimality to characterize the maxmed mechanisms. Like Sprumont [26], I show that maxmed mechanisms continue to be Pareto optimal in the class of mechanisms satisfying anonymity, feasibility, non-bossiness in decision and strategyproofness.

# 2 Model

Consider a 2 agent model with set of agents $N = \{1, 2\}$ and an indivisible object. Each agent $i$ has a private valuation $v_i \geq 0$ for the object. A mechanism is a tuple $(d, \tau)$ such that at any reported profile of valuations $v \in \mathbb{R}_+^N$, each agent $i$ is allocated a transfer $\tau_i(v) \in \mathbb{R}$ and a decision $d_i(v) \in \{0, 1\}$ such that $\sum_{i \in N} d_i(v) \leq 1$. I follow the notation where $d_i(v) = 1$ implies that agent $i$ gets the object, while $d_i(v) = 0$ stands for $i$ not getting the object. Note that I assume that the object may remain unallocated at some profile of reported valuations. Define $w(v)$ to be the agent getting the object at any profile $v$.[7] The utility to agent $i$ with a true valuation of $v_i$ at any reported profile $v' \in \mathbb{R}_+^N$, from a mechanism $(d, \tau)$ is given by $u(d_i(v'), \tau_i(v'); v_i) = v_i d_i(v') + \tau_i(v')$. Let $\forall\, i \neq j \in N$, $\forall\, v \in \mathbb{R}_+^N$, $v_{-i} = v_j$, and define the median of any three real numbers $x, y, z$ as$med\{x, y, z\}$.

Now, a mechanism may have a peculiar allocation decision rule that allocates the object to some agent $j$ if she reports some $\eta > 0$, irrespective of what the other agent $i \neq j$ bids. Alternatively, it may give the object $i$ whenever $j$ reports $\eta$, irrespective of what $i$ bids. Thus, the mechanism may treat an agent $i$ as a dictator, whenever the other agent reports $\eta$. Two examples of such mechanisms are posted-price mechanisms and option-price mechanisms reported in Hagerty and Rogerson [13], Drexl and Kleiner [8] and Shao and Zhou [25]. In my setting, the former implies existence of price $\bar{p}^P$ such that for any valuation profile $v$, $d(v) = (0, 1)$ if and only if $v_1 \leq \bar{p}^P, v_2 \geq \bar{p}^P$; while the latter implies existence of a price $\bar{p}^O$ such that $d(v) = (0, 1)$ if and only if $v_2 \geq \bar{p}^O$. It is easy to see that in both these cases, if $v_2 \in \big[0, \min\{\bar{p}^P, \bar{p}^O\}\big)$, agent 1 becomes a dictator who must be allocated the good irrespective of what she bids. Hagerty and Rogerson [13] interpret this dictatorial behaviour as an "*essentially negative*" feature of a this mechanism.

One of the ways of eliminating such mechanisms from the purview of study is to invoke the axiom of *agent sovereignty*. It is defined in the following manner: every agent $i$ can change the allocation decision by unilaterally changing her report, if the other agent $j$ reports a positive value. This ensures that every agent exerts some influence on the

decision function without imposing any constraint on the transfer function of a mechanism.

[7] I often refer to this agent $w(v)$ as the winner at profile $v$ in the text.

mechanism allocation decision, irrespective of what other agents are bidding.[8]

**Definition 1.** A mechanism $(d, \tau)$ satisfies *agent sovereignty* (AS) if for all $i \neq j \in N$ and all $v \in \mathbb{R}^N_{++}$, $\exists\, v'_i \geq 0$ such that

$$d_i(v) \neq d_i(v'_i, v_j).$$

As shown in Proposition 1, I get agent sovereignty in this paper for free as it is implies by the other axioms defined below.

I also use the following notion of fairness which requires that utility derived from an allocation by any agent be independent of her identity. Thus, any discrimination across agents in terms of utilities received from the mechanism must only be in terms of their valuations for the object. Any mechanism violating this property is likely to be unacceptable in modern societies built upon the inalienable right to equality.

**Definition 2.** A mechanism $(d, \tau)$ satisfies *anonymity in welfare* (AN) if for all $i \in N$, all $v \in \mathbb{R}^N_+$ and all bijections $\pi : N \mapsto N$,

$$u(d_i(v), \tau_i(v); v_i) = u(d_{\pi i}(\pi v), \tau_{\pi i}(\pi v); \pi v_{\pi i}),$$

where $\pi v := \left(v_{\pi^{-1}(k)}\right)_{k=1}^{n}$.

Now, I define a popular strategic axiom in the independent private values setting, strategyproofness, which eliminates the incentive to misreport valuation for each agent by making it a weakly dominant strategy to reveal her true valuation in the ensuing message game.

**Definition 3.** A mechanism $(d, \tau)$ satisfies *strategyproofness* (SP) if $\forall i \in N$, $\forall v_i, v'_i \in \mathbb{R}_+$, $\forall\, v_{-i} \in \mathbb{R}^{N\setminus\{i\}}_+$,

$$u(d_i(v_i, v_{-i}), \tau_i(v_i, v_{-i}); v_i) \geq u(d_i(v'_i, v_{-i}), \tau_i(v'_i, v_{-i}); v_i).$$

Next, I define the axiom of 'non-bossiness in decision' which requires (only) the decision rule in a mechanism to be well-behaved in the sense that no agent is able to influence allocation decision of another agent without changing her own allocation decision.

**Definition 4.** A mechanism $(d, \tau)$ satisfies *non-bossiness in decision* (NBD) if for all $i \in N$, all $v \in \mathbb{R}^N_+$ and all $v'_i \in \mathbb{R}_+$,

$$d_i(v) = d_i(v'_i, v_{-i}) \implies d_j(v) = d_j(v'_i, v_{-i}), \forall\, j \neq i.$$

[8] Similar axioms have been used by Marchant and Mishra [16] and Moulin and Shenker [20].

As noted in Thomson [28], NBD represents a strategic hindrance to collusive practices where agents form groups to misreport their valuations in a coordinated manner so that object allocation decision for any one member changes to her benefit, while others are not worse off.

The following axiom of feasibility requires that the sum of transfers not exceed zero for any profile of valuations and thus, ensures that implementing fair mechanisms do not entail wastage of resources.

**Definition 5.** A mechanism $(d, \tau)$ satisfies *feasibility* if for all $v \in \mathbb{R}^N_+$,

$$\sum_{i \in N} \tau_i(v) \leq 0.$$

In the final axiom below, I present the fairness notion that requires all agents to get a non-negative utility at all possible profiles so that voluntary participation in the mechanism can be ensured.

**Definition 6.** A mechanism $(d, \tau)$ satisfies *individual rationality* (IR) if for all $i \in N$, all $v \in \mathbb{R}^N_+$,

$$v_i d_i(v) + \tau_i(v) \geq 0.$$

To conceptualize the Pareto frontier of any class of mechanisms $\mathcal{S}$, I define a *weak partial order* $\succeq$ on the mechanisms in $\mathcal{S}$ in the following manner. For any two mechanisms $(d, \tau), (d', \tau') \in \mathcal{S}$, let $(d, \tau) \succeq (d', \tau')$ iff for all $i \in N$ and all $v \in \mathbb{R}^N_+$, $u(d_i(v), \tau_i(v); v_i) \geq u(d'_i(v), \tau'_i(v); v_i)$. If in addition, this inequality is strict for some $i$ and some $v$, then I write that $(d, \tau) \succ (d', \tau')$ and say that $(d, \tau)$ *Pareto dominates* $(d', \tau')$. On the other hand, if $u(d_i(v), \tau_i(v); v_i) = u(d'_i(v), \tau'_i(v); v_i)$ for all $i$ and all $v$, then I write that $(d, \tau) \sim (d', \tau')$ and say that $(d, \tau)$ is *Pareto equivalent* to $(d', \tau')$. Finally, I call the class of mechanisms in $\mathcal{S}$ that are not dominated by any other mechanism in $\mathcal{S}$, as the set of *Pareto optimal* mechanisms in $\mathcal{S}$.

# 3 Results

I begin by presenting a well known result which states that the decision rule associated with a strategyproof mechanism must be non-decreasing in one's own reported valuation.[9] More specifically, $\forall\, i$ and $\forall\, v_{-i}$, there exists a finite threshold price $T_i(v_{-i})$ such that: $i$ wins an object if $v_i > T_i(v_{-i})$, and fails to win an object if $v_i < T_i(v_{-i})$.

[9] This result can be found as Proposition 9.27 in Nisan [22] and Lemma 1 in Mukherjee [21].

**Fact 1.** Any mechanism $(d, \tau)$ satisfies SP and AS, if and only if $\forall\, i \in N$ and $\forall\, v_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$, there exist real valued functions $K_i : \mathbb{R}_+^{N\setminus\{i\}} \mapsto \mathbb{R}$ and $T_i : \mathbb{R}_+^{N\setminus\{i\}} \mapsto \mathbb{R}_+$ such that

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > T_i(v_{-i}) \\ 0 & \text{if } v_i < T_i(v_{-i}) \end{cases} \quad \text{and} \quad \tau_i(v) = \begin{cases} K_i(v_{-i}) - T_i(v_{-i}) & \text{if } d_i(v) = 1 \\ K_i(v_{-i}) & \text{if } d_i(v) = 0 \end{cases}$$

**Proof.** This result follows from Proposition 9.27 in Nisan [22] and Lemma 1 in Mukherjee [21]. It is also stated as Fact 1 in Basu and Mukherjee [4]. □

Note that Fact 1 allows for arbitrary tie-breaking in allocation decision of the object at any profile $v \in \mathbb{R}_+^N$ such that $\forall\, i \neq j \in N$ with $v_i \leq T_i(v_j)$. In this paper, *without loss of generality*, I assume a lexicographic tie-breaking rule (as in Sprumont [26]) where the linear order $1 \succ 2$ is used to break ties among agents. That is, for any profile $v$ such that $v_j \leq T_j(v_i)$ for all $j \in N$,

$$d_1(v) = 1 \Longleftrightarrow v_1 = T_1(v_2).$$

Now, I present an result from Basu and Mukherjee [4], which states that any mechanism satisfying AN, AS, NBD and SP; must employ a reserve price $r \geq 0$ such that a top bidder bidding in excess of $r$ wins an object. That is, any such mechanism must employ an allocation rule that is same as that of Vickrey auction with reserve price.

**Fact 2.** If mechanism $(d, \tau)$ satisfies AN, AS, NBD and SP, then there exists an $r \geq 0$ such that for all $i \neq j \in N$ and all $v \in \mathbb{R}_+^N$,

- $T_i(v_j) = \max\{v_j, r\}$ and
- $K_i(v_j) = K(v_j)$ where $K : \mathbb{R}_+^{n-1} \mapsto \mathbb{R}$ is a symmetric functional.

**Proof.** The result follows of from Theorem 1 and Propositions 1 and 2 in Basu and Mukherjee [4]. □

Let $\mathcal{M}$ be the class of mechanisms satisfying AN, NBD, feasibility, IR and SP. I show below that any Pareto optimal mechanism among those in $\mathcal{M}$ must satisfy AS.

**Proposition 1.** *If a mechanism $(d, \tau)$ is Pareto optimal in $\mathcal{M}$, then $(d, \tau)$ satisfies AS.*

**Proof:** Fix any Pareto optimal mechanism $(d, \tau) \in \mathcal{M}$, and suppose that it violates AS. Hence, there exists an $x > 0$ such that the image of the associated threshold function at $x$, $T(x) \notin [0, \infty)$. Now, if $T(x) < 0$, then $d_i(x, x) = 1$ for both $i = 1, 2$ implying a contradiction. On the other hand, if $T(x) = \infty$, then $d_i(x, x) = 0$ for both $i = 1, 2$. Feasibility and IR would then imply that image of the associated $K$ function at $x$, $K(x) = 0$, and so, $u_i(d_i(x, x), \tau_i(x, x); x) = 0, \forall\, i$. Now consider another mechanism $(d', \tau')$ such that for all $v \neq (x, x)$, $(d'_i(v), \tau'_i(v)) = (d_i(v), \tau_i(v))$ for all $i$, while $d'_1(x, x) = 1, d'_2(x, x) = 0$ and $\tau'_1(x, x) = -\frac{x}{2}, \tau'_2(x, x) = \frac{x}{2}$. It is easy to see that $(d', \tau') \in \mathcal{M}$. Further, for each

$i = 1, 2$, $u_i(d'_i(v), \tau'_i(v); v_i) - u_i(d_i(v), \tau_i(v); v_i)$ is positive if $v = (x, x)$, or else it is zero. This implies that $(d', \tau') \succ (d, \tau)$, which is a contradiction to Pareto optimality of $(d, \tau)$. Hence, the result follows. $\square$

Now I define a class of maxmed mechanisms $\Gamma$.

**Definition 7.** *For any $r \geq 0$, let $(d^r, \tau^r)$ be a maxmed mechanism such that for any $i \neq j \in N$, and any $v \in \mathbb{R}^N_+$,*

- $d_i(v) = \begin{cases} 1 & \text{if } v_i > \max\{v_j, r\} \\ 0 & \text{if } v_i < \max\{v_j, r\} \end{cases}$
- $\tau_i(v) = \begin{cases} med\{0, v_j - r, r\} - \max\{v_j, r\} & \text{if } d_i = 1 \\ med\{0, v_j - r, r\} & \text{if } d_i = 0 \end{cases}$
- $d_1(v) = 1, \forall\, v \ni v_i = v_j \geq r.$

*Let $\Gamma := \{(d^r, \tau^r)\}_{r \geq 0}$ be the class of all possible maxmed mechanisms.*

The main result of this paper is the following complete characterization of $\Gamma$. I show below that $\Gamma$ is the unique class of Pareto optimal mechanisms in $\mathcal{M}$.

**Theorem 1.** $\Gamma$ *is the unique set of Pareto optimal mechanisms in $\mathcal{M}$.*

**Proof:**
`Necessity.` Fix any Pareto optimal mechanism $(d, \tau)$ in $\mathcal{M}$, any $i \neq j \in N$, and any $v \in \mathbb{R}^N_+$. By Proposition 1, $(d, \tau)$ satisfies AS, and so, by Fact 2, there exists an $r \geq 0$ such that the threshold function associated with $(d, \tau)$ is $T(x) = \max\{x, r\}$ for all $x \geq 0$. Now, fix any $y < r$, and note that: if $v_i = v_j = y$, feasibility and IR imply that (**i**) $K(y) = 0$. Now fix any profile $v$ such that for any $i \neq j$, $v_i \geq 2r$ and $v_j < r$. By feasibility and IR, $0 \leq K(v_i) + K(v_j) \leq r$. So, by (**i**), I can infer that (**ii**) $0 \leq K(x) \leq r, \forall x \geq 2r$. If there exists an $x' \geq 2r$ such that $K(x') \in [0, r)$, then I can construct another mechanism $(d', \tau')$ such that,

- $(d'(v), \tau'(v)) = (d(v), \tau(v))$ for all $v$ such that for any $i \neq j$, $v_i \neq x'$ and $v_j \neq x'$,
- $(d'(v), \tau'(v)) = (d(v), \tau(v))$ for all $v$ such that for any $i \neq j$, $v_i = x'$ and $v_j \geq r$,
- $d'(v) = d(v)$ for any $v$ such that for any $i \neq j \in N$ with $v_i = x'$ and $v_j < r$,
- for any $v$ such that there exists $i \neq j \in N$ with $v_i = x'$ and $v_j < r$, $\tau'_j(v) = r$ and $\tau'_i(v) = -r$.

It is easy to see that for any $i \neq j \in N$ and for any $\hat{v} \in \mathbb{R}^N_+$: if $\hat{v}_i = x', \hat{v}_j < r$ then $u_j(d'_j(\hat{v}), \tau'_j(\hat{v}); \hat{v}_j) = r > K(x') = u_j(d_j(\hat{v}), \tau_j(\hat{v}); \hat{v}_j)$ and $u_i(d'_i(\hat{v}), \tau'_i(\hat{v}); \hat{v}_i) = x' -$

$r = u_i(d_i(\hat{v}), \tau_i(\hat{v}); \hat{v}_i)$, or else $u_h(d'_h(\hat{v}), \tau'_h(\hat{v}); \hat{v}_h) = u_h(d_h(\hat{v}), \tau_h(\hat{v}); \hat{v}_h), \forall\, h \in N$. Thus, $(d', \tau') \succ (d, \tau)$, which implies that $(d, \tau)$ is not Pareto optimal, and hence, I get a contradiction. Therefore, by (**ii**), I can infer that (**iii**) $K(x) = r, \forall x \geq 2r$.
Now consider any $z \in [r, 2r)$, and consider a profile $v$ with $v_i \geq 2r$ and $v_j = z$. It is easy to see that (**iii**), feasibility and IR imply that $0 \leq K(z) \leq z - r$. Now, if there exists a $z' \in [r, 2r)$ such that $K(z') \in (0, z' - r)$, I can construct another mechanism $(d'', \tau'')$ such that,

- $(d''(v), \tau''(v)) = (d(v), \tau(v))$ for all $v$ such that for any $i \neq j$, $v_i \neq z'$ and $v_j \neq z'$,
- $(d''(v), \tau''(v)) = (d(v), \tau(v))$ for all $v$ such that for any $i \neq j$, $v_i = z'$ and $v_j < 2r$,
- $d''(v) = d(v)$ for any $v$ such that for any $i \neq j \in N$ with $v_i = z'$ and $v_j \geq 2r$,
- for any $v$ such that there exists $i \neq j \in N$ with $v_i = z'$ and $v_j \geq 2r$, $\tau''_j(v) = -r$ and $\tau''_i(v) = r$.

As before, for any $v \in \mathbb{R}^N_+$: if $v_i = z', v_j \geq 2r$ then $u_j(d''_j(v), \tau''_j(v); v_j) = v_j - z' + z' - r > v_j - z' + K(z') = u_j(d_j(v), \tau_j(v); v_j)$ and $u_i(d'_i(v), \tau'_i(v); v_i) = r = u_i(d_i(v), \tau_i(v); v_i)$, or else $u_h(d'_h(\hat{v}), \tau'_h(\hat{v}); \hat{v}_h) = u_h(d_h(\hat{v}), \tau_h(\hat{v}); \hat{v}_h), \forall\, h \in N$. Hence, as before, $(d'', \tau'') \succ (d, \tau)$ - which is a contradiction to the supposition of $(d, \tau)$ being Pareto optimal. Thus, I get that for any $x \geq 0$, the $K(.)$ function associated with $(d, \tau)$ must have an image as follows;

$$K(x) = \begin{cases} 0 & \text{if } x < r \\ x - r & \text{if } x \in [r, 2r) \\ r & \text{if } x \geq 2r. \end{cases}$$

Thus, $K(x) = med\{0, x - r, r\}, \forall\, x \geq 0$, and so the result follows.

`Sufficiency.` Fix any $r \geq 0$ and any mechanism $(d^r, \tau^r) \in \Gamma$. By Fact 1, it is easy to see that $(d^r, \tau^r)$ satisfies AN and SP because, for all $x > 0$, the associated threshold function $T(x) = \max\{x, r\}$ and $K(x) = med\{0, x - r, r\}$. Hence, it is easy to see that $(d^r, \tau^r)$ satisfies NBD. To see that $(d^r, \tau^r)$ satisfies IR, note that for any $i$ and $v$, $d_i(v) = 0 \implies \tau_i(v) = 0$, while $d_i(v) = 1 \implies |\tau_i(v)| \leq v_i$. Further, note that $\sum_{h \in N} K(v_{-h}) = 0$ whenever $\sum_{h \in N} d_h(v) = 0$, and if there exists an $i \in N$ with $d_i(v) = 1$, then for $i \neq j$:

$$\sum_{h \in N} \tau_h(v) = \begin{cases} 2r - v_j & \text{if } v_i \geq v_j \geq 2r \\ 0 & \text{if } v_i \geq 2r > v_j \geq r \\ 0 & \text{if } v_i \geq 2r > r > v_j \\ v_i - 2r & \text{if } 2r > v_i \geq v_j \geq r \\ v_i - 2r & \text{if } 2r > v_i \geq r > v_j. \end{cases}$$

Therefore, $(d^r, \tau^r)$ satisfies feasibility.

Finally, to prove that $(d^r, \tau^r)$ is Pareto optimal, suppose the contrapositive - that is, suppose that there exists a $(d, \tau) \notin \Gamma$ such that $(d, \tau) \succ (d^r, \tau^r)$. Then, by the proof of necessity, there exists an $r' \geq 0$ and a maxmed mechanism $(d^{r'}, \tau^{r'}) \in \mathcal{M}$ such that $(d^{r'}, \tau^{r'}) \succeq (d, \tau)$, and so, $(d^{r'}, \tau^{r'}) \succ (d^r, \tau^r)$. Therefore, I can infer that $r \neq r'$. Now if $r > r'$, then consider a profile $v$ such that $v_i > v_j > 2r$, and note that $u_j(d_j^r(v), \tau_j^r(v); v_j) := r > r' = u_j(d_j^{r'}(v), \tau_j^{r'}(v); v_j)$, which contradicts $(d^{r'}, \tau^{r'}) \succ (d^r, \tau^r)$. Similarly, if $r < r'$, then there exists a profile $\hat{v}$ such that $\hat{v}_i > \max\{r', 2r\} \geq \min\{r', 2r\} > \hat{v}_j > r$, and note that $u_i(d_i^r(\hat{v}), \tau_i^r(\hat{v}); \hat{v}_i) = \hat{v}_i - \hat{v}_j + \hat{v}_j - r = \hat{v}_i - r > \hat{v}_i - r' = u_i(d_i^r(\hat{v}), \tau_i^r(\hat{v}); \hat{v}_i)$, which again contradicts $(d^{r'}, \tau^{r'}) \succ (d^r, \tau^r)$. Hence, the result follows. $\square$

The Theorem 1 above establishes the maxmed mechanisms as the unique Pareto optimal mechanisms in $\mathcal{M}$. Now, it is easy to see that no maxmed mechanism can be Pareto dominated by any other mechanism that violates IR. This fact implies that maxmed mechanisms continue to be Pareto optimal in the class, say $\hat{\mathcal{M}}$, of anonymous, feasible, non-bossy in decision and strategyproof mechanisms. However, one may construct mechanisms in $\hat{\mathcal{M}}$ that are Pareto undominated by any maxmed mechanism. Hence, $\Gamma$ is no longer the unique Pareto optimal class of mechanisms in $\hat{\mathcal{M}}$. This result is reported below as a corollary of Theorem 1 without proof.[10]

**Corollary 1.** $\Gamma$ *is Pareto optimal in* $\hat{\mathcal{M}}$*.*

# 4 Discussion

As mentioned earlier, the characterization of maxmed mechanisms of this paper differs from that of Sprumont [26] in terms of number of agents considered (which in this paper is 2) and the substitution of the no-envy axiom by non-bossiness.[11] To check the relationship between no-envy and non-bossiness; consider the mechanism $(d, \tau)$ where $T(x) = x$ and $K(x) = 2x$ for all $x \geq$. This mechanism satisfies NBD since the object is allocated at all profiles, while it violates no-envy at profile $(10, 2)$ where $\tau_1(10, 2) = 2$ and $\tau_2(10, 2) = 20$ (implying that agent 1 prefers the allocation bundle of agent 2). Thus, NBD does not imply no-envy. Similarly, consider a mechanism $(d', \tau')$ where the associated functions are $T'(x) = \begin{cases} x & \text{if } x \leq 10 \\ 2x & \text{if } x > 10 \end{cases}$ and $K'(x) = 0$ for all $x \geq 0$. It is easy to see that this mechanism satisfies no-envy. However, $d'(12, 10) = (1, 0)$ but $d'(12, 12) = (0, 0)$, and $(d', \tau')$ violates NBD. Thus, no-envy does not imply NBD. Thus, it follows that no-envy and NBD are logically independent axioms.

[10] Note that Proposition 1 holds true for $\hat{\mathcal{M}}$ too.

[11] Formally, a mechanism $(d, \tau)$ is said to satisfy no-envy if for any $i \neq j \in N$ and any $v$, $u_i(d_i(v), \tau_i(v); v_i) \geq u_i(d_j(v), \tau_j(v); v_i)$.

## 5 Conclusion

I present a new characterization of maxmed mechanisms that of Sprumont [26], by substituting the axiom of no-envy with axiom of non-bossiness in decision. I show that in a simple two agent setting, maxmed mechanisms are the only Pareto optimal mechanisms in the class of anonymous, feasible, individually rational, non-bossy in decision, and strategyproof mechanisms. Extension of this characterization to the general $n$ agent setting, or to the multiple object setting is a difficult exercise. I leave these questions for future research.

## References

[1] K. Apt, V. Conitzer, M. Guo, and E. Markakis. Welfare undominated Groves mechanisms. In *International Workshop on Internet and Network Economics*, pages 426–437, 2008.

[2] I. Ashlagi and S. Serizawa. Characterizing Vickrey allocation rule by anonymity. *Social Choice and Welfare*, 38:531–542, 2012.

[3] E. Athanasiou. A solomonic solution to the problem of assigning a private indivisible good. *Games and Economic Behavior*, 82:369–387, 2013.

[4] R. Basu and C. Mukherjee. Characterization of Vickrey auction with reserve price for multiple objects. *Review of Economic Design*, pages 1–28, 2022.

[5] E. Clarke. Multipart pricing of public goods. *Public choice*, 11:17–33, 1971.

[6] G. de Clippel, V. Naroditskiy, and A. Greenwald. Destroy to save. In *Proceedings of the 10th ACM conference on Electronic commerce*, pages 207–214, 2009.

[7] M. Drexl and A. Kleiner. Optimal private good allocation: The case for a balanced budget. *Games and Economic Behavior*, 94:169–181, 2015.

[8] M. Drexl and A. Kleiner. Optimal private good allocation: The case for a balanced budget. *Games and Economic Behavior*, 94:169–181, 2015.

[9] M. P. Goswami, M. Mitra, and A. Sen. Strategy proofness and pareto efficiency in quasilinear exchange economies. *Theoretical Economics*, 9:361–381, 2014.

[10] T. Groves. Incentives in teams. *Econometrica*, 41:617–631, 1973.

[11] M. Guo and V. Conitzer. Worst-case optimal redistribution of VCG payments in multi-unit auctions. *Games and Economic Behavior*, 67:69–98, 2009.

[12] Mingyu Guo and Vincent Conitzer. Optimal-in-expectation redistribution mechanisms. *Artificial Intelligence*, 174:363–381, 2010.

[13] K. Hagerty and W. Rogerson. Robust trading mechanisms. *Journal of Economic Theory*, 42:94–107, 1987.

[14] K. Hashimoto and H. Saitoh. Strategyproof and anonymous rule in queueing problems: a relationship between equity and efficiency. *Social Choice and Welfare*, 38:473–480, 2012.

[15] Y. Long, D. Mishra, and T. Sharma. Balanced ranking mechanisms. *Games and Economic Behavior*, 105:9–39, 2017.

[16] D. Mishra and T. Marchant. Mechanism design with two alternatives in quasi-linear environments. *Social Choice and Welfare*, 44:433–455, 2015.

[17] D. Mishra and A. Quadir. Non-bossy single object auctions. *Economic Theory Bulletin*, 2014. Forthcoming.

[18] H. Moulin. Almost budget-balanced VCG mechanisms to assign multiple objects. *Journal of Economic theory*, 144:96–119, 2009.

[19] H. Moulin. Auctioning or assigning an object: some remarkable VCG mechanisms. *Social Choice and Welfare*, 34:193–216, 2010.

[20] H. Moulin and S. Shenker. Strategyproof sharing of submodular costs:budget balance versus efficiency. *Economic Theory*, 18:511–533, 2001.

[21] C. Mukherjee. Fair and group strategy-proof good allocation with money. *Social Choice and Welfare*, 42:289–311, 2014.

[22] N. Nisan. *Algorithmic game theory*. Cambridge University Press, 2007.

[23] S. Ohseto. Characterizations of strategy-proof and fair mechanisms for allocating indivisible goods. *Economic Theory*, 29(1):111–121, 2006.

[24] M. Satterthwaite and H. Sonnenschein. Strategy-proof allocation mechanisms at differentiable points. *The Review of Economic Studies*, 48:pp. 587–597, 1981.

[25] R. Shao and L. Zhou. Optimal allocation of an indivisible good. *Games and Economic Behavior*, 100:95–112, 2016.

[26] Y. Sprumont. Constrained-optimal strategy-proof assignment: Beyond the Groves mechanisms. *Journal of Economic Theory*, 148:1102–1121, 2013.

[27] Lars-Gunnar Svensson. Strategy-proof allocation of indivisible goods. *Social Choice and Welfare*, 16:557–567, 1999.

[28] W. Thomson. Non-bossiness. *Social Choice and Welfare*, 47:665–696, 2016.

[29] W. Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of finance*, 16:8–37, 1961.