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STRATEGYPROOF MULTIDIMENSIONAL MECHANISM DESIGN WITHOUT UNIT DEMAND

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Abstract. We consider the heterogeneous object auction problem where buyers have private additive valuations and non-unit demand. We completely characterize the class of strategyproof and agent sovereign mechanisms. Further, we introduce a notion of continuity, and show that every continuous, agent sovereign, anonymous and strategyproof mechanism must be efficient. We find that the only mechanism satisfying these properties is equivalent to operating simultaneous second price auctions for each object - as was done by New Zealand government in allocating license rights to use of radio spectrum in 1990. Finally, we present a complete characterization of simultaneous second price auctions with object specific reserve prices, in terms of these properties and a weak *non-bossiness* restriction.

JEL classification: D44; D47; D63; D71; D82

Keywords: Mechanism design, Heterogeneous objects auction, Non-unit demand, Strategyproofness, Pivotal mechanism.

1. Introduction

We consider the problem where a planner wishes to sell m heterogeneous indivisible objects to n buyers who have private additive valuations. The planner is unaware of buyers' valuation, and each buyer has linear preferences over objects and money. This problem encompasses many real life applications ranging from spectrum auction to airport landing rights allocation. All such exercises invariably require reporting of personal valuations by interested buyers. How to execute such a sale in a manner that all buyer report truthfully, is an important problem in mechanism design.

In this paper, we adopt the most robust notion of truthful reporting, *strategyproofness*, which requires that all participants find it optimal to report their true valuations, irrespective of what all other participants choose to do. As is well known, the remarkable advantage of using this notion of truthful revelation of valuations is that no prior distributional assumptions are required to justify policies implementing strategyproof mechanisms. More specifically we address the question: *what are the strategyproof mechanisms for a heterogeneous object allocation problem?*

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Note that addressing this question in its most general form is a difficult task as the domain of valuations in our setting is a set of matrices with positive elements. Fortunately, the policy context of mechanism design requires us to focus on mechanisms that would be acceptable in a democratic society. Such mechanisms are most aptly described in the following quote from the Ministry of Business, Innovation & Employment, Government of New Zealand report [25].[1]

> *"Requirements of the* [spectrum] *allocation process can be summarised as follows:*
>
> - *It is a market based allocation process.*
> - *It allocates at a price that reflects the market value of the product.*
> - *It allocates in a manner that is competitively neutral and transparent."*

We accommodate this policy context by focusing on well-behaved strategyproof mechanisms satisfying two crucial properties: *agent sovereignty* and *anonymity*.

The former property essentially requires that the object allocation decision for any buyer i depend non-trivially on i's valuations in all possible states of nature. The latter property requires that no buyer should get a preferential treatment based on her social identity. Note that any mechanism violating agent sovereignty would fail to always allot objects at true "market value", as at some states of nature it would have to ignore valuations of at least one buyer irrespective of how high they are. On the other hand, any non-anonymous mechanism would fail to be "competitively neutral".

We first provide a complete characterization of agent sovereign strategyproof mechanisms, which holds true regardless of the number of distinct objects and the number of buyers. Next, we look to identify a strategyproof mechanisms that are agent sovereign and anonymous. However, the dimensionality problem inherent in our setting, makes this an intractable problem. To get around this complication, we impose a regularity condition of *continuity* on the class of mechanisms, and characterize the class of continuous agent sovereign strategyproof mechanisms satisfying anonymity. Remarkably, we find that any such mechanism that never leaves an object unsold must be decision efficient. More specifically, the only mechanism satisfying these properties is equivalent to operating simultaneous sealed bid second price auctions for each object; which, incidentally, was the chosen method for allocation of three distinctly defined cellular (management rights) tenders in New Zealand's spectrum auction of 1990.[2] Thus, our paper can also be motivated as an axiomatic justification to this method of spectrum allocation.

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[1]See point 8 in page 3 in Milgrom [17].

[2]As discussed in Mueller [23], a British-American consulting firm National Economic Research Associates (NERA) recommended this specific allocation procedure. These tenders brought in roughly 80% of the total revenue earned in the first four rounds of radio license auctions. We discuss this issue in greater detail in the Remark 3.

Finally, we use an additional axiom, *non-bossiness in decision*, to completely characterize the class of continuous, agent sovereign, anonymous and non-bossy mechanisms that *allow* the possibility of some objects remaining unsold. We find that any such mechanism is equivalent to operating simultaneous sealed bid second price auctions for each available object with *varying* reserve prices across objects. This non-bossiness axiom is a modified version of the conventional non-bossiness axiom of Satterthwaite and Sonnenschein [27], and requires that no buyer be able to affect the allocation decision of another buyer without affecting her own allocation decision. As noted by Thomson [30], this axiom, when coupled with strategyproofness, embodies strategic restrictions that discourage collusive practices.

The paper is organized as follows. The next section 2 contains the literature review, section 3 contains model description and relevant definitions. Section 4 contains the results, section 5 presents a discussion of our results, while section 6 presents the conclusion. Finally, section 7 is the Appendix where proofs and independence of axioms is presented.

2. Relation to literature

Note that our paper assumes that buyers have private additive valuations for the heterogeneous objects up for sale. Some notable papers that analyze sale of heterogeneous objects in the private value setting are: Ausubel [2], Ausubel, Cramton, Pycia, Rostek and Weretka [3], Ausubel and Milgrom [4], Demange, Gale and Sotomayer [9], de Vries, Schummer and Vohra [7], Gul and Stacchetti [12], Mishra and Parkes [19], and Kazumura, Mishra, and Serizawa [14].

Ausubel, Cramton, Pycia, Rostek and Weretka [3] compare first price and second price auction in terms of the degree of inefficiency in the resultant Bayes Nash equilibrium, and show that expected revenue rankings are ambiguous. Ausubel [2] provides a new interpretation of Walrasian equilibrium by describing a dynamic auction procedure where strategic bidders reveal their preferences over a single price path, and the corresponding Walrasian equilibrium outcome is achieved. Ausubel and Milgrom [4] present a collection of ascending combinatorial bidding auctions where valuations are additive and truthful reporting is a Nash equilibrium.

Demange, Gale and Sotomayer [9] studies a setting where bidders have unit demand, and describes two dynamic auctions which attain (one exactly and the other approximately) the minimum price equilibrium of a sealed bid auction for heterogeneous objects. Similar unit demand settings have also been analyzed in private values framework by Gale and Shapley [12], Shapley and Shubik [28]. Gul and Stacchetti [12] extend Demange, Gale and Sotomayer [9] to general combinatorial auctions allowing for valuations to be gross substitutes in the sense of Kelso and Crawford [15]. They construct an ascending price auction that achieves the minimum price equilibrium, and show that ascending auctions can implement efficient strategy-proof outcomes only in a unit demand setting.

de Vries, Schummer and Vohra [7] further generalize this setting to unrestricted valuations, and constructs an ascending auction which leads to VCG (Vickrey [31], Clarke [5], Groves [11]) outcome prices when valuations satisfy a 'submodularity' property (that is a weaker restriction on valuations than gross substitutes). Mishra and Parkes [19] relax the de Vries, Schummer and Vohra [7] definition of ascending price auctions suitably, to construct ascending price auctions which maintain single path but attain VCG outcomes for a larger class of valuation functions in ex-post Nash equilibrium.

Unlike all these papers, the main objective of this paper is not to construct or compare auction algorithms. Instead, this paper primarily focusses on the standard direct mechanism design question: *what are the strategyproof mechanisms for auctioning heterogeneous objects when buyers have additive valuations and non-unit demand?*

To our knowledge, the only paper that considers a similar question is Kazumura, Mishra and Serizawa [14] (henceforth, referred to as [KMS]). They consider mechanisms, which always allot all available objects in a heterogeneous object setting with buyers having unit demand and (possibly) non-quasilinear preferences. They focus on the MWEP (minimum Walrasian equilibrium price) mechanism which, at any reported valuation profile, makes allotment and price decision as dictated by the minimum Walrasian equilibrium price vector and its corresponding equilibrium allocation at that profile. They establish a remarkable result which shows that; on a variety of rich domains, MWEP mechanism is the unique ex-post revenue maximizing mechanism, among all possible *desirable* strategyproof mechanisms that sell all objects and charge non-negative price at all profiles.[3]

Our results are independent of those in KMS [14], as we allow for more general non-unit demand preferences, but use a more restrictive quasilinear utility setting. Furthermore, our setting allows for objects to remain unsold, and so, does not rule out usage of reserve prices. In fact, we present an axiomatic justification (in Theorem 5) to the usage of different reserve prices for different objects.

Some other notable papers that have posed similar mechanism design questions in the context of heterogeneous object setting are: Alkan, Demange and Gale [1], and Pápai [24]. Alkan, Demange and Gale [1] eschews strategic considerations and analyzes fair distribution of objects and money. In particular, this paper focusses on the *envy-free* allocations where every buyer

[3]KMS [14] consider a mechanism to be *desirable* if and only if it satisfies strategyproofness, ex-post individual rationality, equal treatment of equals, and it sells all objects at all reported valuation profiles.

likes her allocation at least as much as another buyer, and show that such allocations must also be Pareto efficient. Pápai [24] looks at strategyproof mechanisms that generate envy-free allocations. She first notes an impossibility where valuation functions are unrestricted, and then identifies a subset of VCG mechanisms that generate envy-free allocations when valuations are superadditive.

Another strand of literature that links to our work is the analysis of monopoly pricing with a single buyer with multidimensional private information. Two notable papers that analyze the problem of expected revenue maximization in a setting where there are several heterogeneous objects need to be sold to a single buyer with additive valuations are: Manelli and Vincent [16] and Rochet and Choné [26]. Both papers address this question under specified distributional assumptions on the multidimensional private information. Our paper, however, adopts a direct mechanism approach which focusses on eliciting true valuation at all states of nature.

We are unaware of any other paper that analyzes heterogeneous object sales from a mechanism design perspective in a private additive valuation setting with multiple buyers and non-unit demand.

3. Model

Fix any $m \geq 2$ and $n \geq 1$. Consider indivisible objects in $M = \{1, 2, \ldots, m\}$ to be sold to buyers in $N = \{1, \ldots, n\}$, where each buyer i has a positive private valuation $v_i^k \in \mathbb{R}_{++}$ for each object $k \in M$. Let $v_i := (v_i^k)_{k \in M}$ denote a typical valuation vector of any buyer i. Let $V_i := \mathbb{R}^m_{++}$ be the set of all such valuation vectors, and let $\mathcal{V} := \Pi_{i \in N} V_i$ be the set of all possible valuation profiles, where each profile is a $n \times m$ matrix. For each object k and each buyer i, define the variable $d_i^k \in \{0, 1\}$ where $d_i^k = 1$ if and only if i is sold the object k. Let $d_i \in \{0, 1\}^m$ be the decision vector assigned to any buyer i. Define an $n \times m$ decision matrix $d := \left(d_i^k\right)$ such that: (i) its elements are either 0 or 1, (ii) each of its n rows are points in $\{0, 1\}^m$, and (iii) elements of each column sum up to 1. Define $\mathcal{D}$ to be the set of all such decision matrices. Each buyer i is assumed to have quasilinear preference over $\mathcal{D} \times \mathbb{R}^N_+$ such that utility of each pair of decision matrix and price vector (d, p) is given by $u((d, p); v_i) := d_i^T v_i - p_i$. Further, for any $t \in \mathbb{N}$, let $\mathbf{1}^t := (1, 1, \ldots, 1) \in \mathbb{R}^t$, and $\mathbf{0}^t := (0, 0, \ldots, 0) \in \mathbb{R}^t$.[4]

A direct mechanism employed to accomplish this allocation exercise must be a function:

$$\mu : \mathcal{V} \longrightarrow \mathcal{D} \times \mathbb{R}^N_+.$$

[4]Unless specified otherwise, all vectors are considered to be column vectors in our paper. Therefore, any valuation profile $v = (v_1^T, v_2^T, \ldots, v_n^T)$.

For direct mechanism μ, let $d^{\mu}(v)$ and $p^{\mu}(v)$ denote the allotment decision matrix and price vector, respectively; corresponding to any valuation profile $v \in \mathcal{V}$. Let $d_i^{\mu}(v)$ and $p_i^{\mu}(v)$ be the decision vector assigned to i and the price charged to i, respectively, by mechanism μ. Further, to economize on the use of notation, we drop the superscript μ, whenever this does not create any confusion.

For any $v \in \mathcal{V}$, and any $i \in N$, let $O_i(v) := \{k \in M | d_i^k(v) = 1\}$. Further, we define for any $k \in M$ and any $i \in N$, $d_i^{-k} := (d_i^1, \dots, d_i^{k-1}, d_i^{k+1}, \dots, d_i^m)$. Similarly, we define for any $i \in N$ and any $v \in \mathcal{V}$, $v_{-i} := (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$. Also, for any $i \in N$, $k \in M$ and any $v \in \mathcal{V}$, we define $v_i^{-k} := (v_i^1, \dots, v_i^{k-1}, v_i^{k+1}, \dots, v_i^m)$, and $v_{-i}^k := (v_1^k, \dots, v_{i-1}^k, v_{i+1}^k, \dots, v_n^k)$. Finally, for any $k \neq l \in M$ and any $i \neq j \in N$, define $d_i^{-k-l} := (d_i^1, \dots, d_i^{k-1}, d_i^{k+1}, \dots, d_i^{l-1}, d_i^{l+1}, \dots, d_i^m)$, $v_{-i-j} := (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1} \dots, v_n)$, and $v_i^{-k-l} := (v_i^1, \dots, v_i^{k-1}, v_i^{k+1}, \dots, v_i^{l-1}, v_i^{l+1} \dots, v_i^m)$.[5]

In this paper, we focus on *regular* direct mechanisms which satisfy two specific properties:

(i): any buyer who is not allotted any object pays zero price, and

(ii): for any buyer i, any object k and any valuation v, there exists an $\epsilon_i^k(v) > 0$ such that for any $w_i^k \in (v_i^k - \epsilon_i^k(v), v_i^k + \epsilon_i^k(v))$,

$$d_i^{-k}(v) = d_i^{-k}((w_i^k, v_i^{-k}), v_{-i}).$$

The property (i) is a natural restriction in any publicly organized sale exercise. The property (ii), however, is more technical in nature. It requires that the decision function employed in the mechanism be well-behaved in the following sense. At any profile v, any buyer i should be able to change her valuation for object k *slightly*, without affecting her allotment decisions for other objects. This property endows a degree of smoothness to the mechanisms which ensures analytical tractability of the research question.

Now, we can define the two of the three main axioms that we employ in this paper. The first axiom of *strategyproofness* is a strategic one, which requires that truth-telling be an optimal message while participating in a direct mechanism, irrespective of valuation reported by all other buyers. This is one the most popular strategic axioms used in mechanism design. The second

[5]In a setting where $m = n = 3$, if we consider a valuation profile $v := \begin{pmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{pmatrix}$, then:

$$v_1 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \ v^2 = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}, \ v_{-1} = \begin{pmatrix} 3 & 6 & 9 \\ 4 & 8 & 12 \end{pmatrix}, \ v^{-2} = \begin{pmatrix} 2 & 6 \\ 3 & 9 \\ 4 & 12 \end{pmatrix}.$$

axiom *agent sovereignty* is a fairness axiom, which describes the individual right of each buyer to target and win any particular object by reporting large enough valuation for it, irrespective of what other buyers are reporting. Any violation of this axiom may lead to situations where some buyer's allotment decision for an object is independent of her valuation for that object.[6]

Definition 1. A mechanism μ satisfies *strategyproofness* (SP) if $\forall\, i \in N$, $\forall\, v, v' \in \mathcal{V}$ such that $v_{-i} = v'_{-i}$,

$$u(\mu_i(v); v_i) \geq u(\mu_i(v'); v_i).$$

Definition 2. A mechanism μ satisfies *agent sovereignty* (AS) if $\forall i \in N$, $\forall k \in M$, $\forall v_i^{-k} \in \mathbb{R}^{m-1}_{++}$, and $\forall\, v_{-i} \in \Pi_{j\neq i} V_j$, there exist $v_i^k, w_i^k > 0$ such that:

$$d_i^k((v_i^k, v_i^{-k}), v_{-i}) \neq d_i^k((w_i^k, v_i^{-k}), v_{-i}).$$

Let $\Gamma_{M,N}$ be the set of mechanisms that satisfy SP and AS.

We now define an additional fairness axiom of *anonymity* that is crucially important for public decision making procedures. It requires that utility derived from an allocation by any buyer be independent of her identity. Any mechanism violating this property is highly unlikely to be acceptable in a democratic society.

Definition 3. A mechanism μ satisfies *anonymity in welfare* (AN) if $\forall\, i \in N$, $\forall\, v \in \mathcal{V}$ and all bijections $\pi : N \mapsto N$,

$$u(\mu_i(v); v_i) = u(\mu_{\pi i}(\pi v); (\pi v)_{\pi i})$$

where $\pi v := \left(v_{\pi^{-1}(t)}\right)_{t=1}^{n}$.

4. Results

4.1. Single buyer case.

This section presents results for the simplest case where there is only one buyer. Therefore, the set of all possible decisions $\mathcal{D} = \{0,1\}^m$ and $\mathcal{V} = \mathbb{R}^m_{++}$. It is easy to see that in this simple setting SP is equivalent to incentive compatibility. The following result characterizes the class of incentive compatible mechanisms.

Theorem 1. *A mechanism $\mu \in \Gamma_{M,1}$ if and only if for any $v \in \mathcal{V}$, there exist positive real numbers $\{T^k\}_{k\in M}$ such that for all $v \in \mathcal{V}$ and all $k \in M$:*

[6]Similar axioms have been used by Marchant and Mishra [18], Moulin [21] and Moulin and Shenker [22]. Moulin [21] mentions the agent sovereignty axiom to be "*reminiscent of the citizen sovereignty of classical social choice.*" In our setting, this axiom rules out peculiar mechanisms like those where all objects are always allotted to a specific buyer; or those where a buyer never gets sold any object.

(1) $d^k(v) = \begin{cases} 1 & \textit{if } v^k > T^k \\ 0 & \textit{if } v^k < T^k \end{cases}$ *, and*

(2) $p(v) = \sum_{k:d^k(v)=1} T^k.$

Proof of Necessity: Consider any two profiles $v_x, v_y \in \mathcal{V}$, and fix any object $k \in M$. SP implies that $u(d(v_x), p(v_x); v_x) \geq u(d(v_y), p(v_y); v_x)$ and $u(d(v_x), p(v_x); v_y) \leq u(d(v_y), p(v_y); v_y)$. This implies that $p(v_y) - p(v_x) \geq v_x^T[d(v_y) - d(v_x)]$ as well as $p(v_y) - p(v_x) \leq v_y^T[d(v_y) - d(v_x)]$. Therefore, we get the following inequality:

$$v_x^T[d(v_y) - d(v_x)] \leq p(v_y) - p(v_x) \leq v_y^T[d(v_y) - d(v_x)] \tag{4.1}$$

There are several interesting implications of (4.1). The most obvious is, (**a**) $d(v_y) = d(v_x) \implies p(v_y) = p(v_x)$. This implies that for any $v \in \mathcal{V}$, $p(v) \in \{\bar{p}^d\}_{d\in\mathcal{D}}$, where for any $d \in \mathcal{D}$, $\bar{p}^d$ is the price to be charged when the buyer is assigned the decision vector d. Further, (4.1) implies that (**b**) $v_x^T[d(v_y) - d(v_x)] \leq v_y^T[d(v_y) - d(v_x)]$. This implies that for any $d \in \mathcal{D}$, and any $k \in M$, $v_x^{-k} = v_y^{-k} \implies v_x^k[d^k(v_y) - d^k(v_x)] \leq v_y^k[d^k(v_y) - d^k(v_x)]$. Thus, we can infer that for all $k \in M$, and all $v, v' \in \mathbb{R}^m_{++}$ with $v^{-k} = v'^{-k}$,

$$v^k > v'^k \implies d^k(v) \geq d^k(v').$$

Therefore, by AS, for any $k \in M$ and any $d^{-k} \in \{0,1\}^{m-1}$, there exists a threshold function $T^{k,d^{-k}} : \mathbb{R}^{m-1}_{++} \mapsto \mathbb{R}_{++}$ such that for all $v \in \mathcal{V}$:[7]

$$d^k(v) = \begin{cases} 1 & \text{if } v^k > T^{k,d^{-k}}(v^{-k}) \\ 0 & \text{if } v^k < T^{k,d^{-k}}(v^{-k}). \end{cases}$$

Now, there may be two possible cases:

- **(i):** for all k and all v, $T^{k,d^{-k}}(v^{-k})$ values depend on d^{-k} but d^{-k} does not depend on v^k, and
- **(ii):** for all k and all v, the $T^{k,d^{-k}}(v^{-k})$ values are independent of the values taken by d^{-k}.

We consider each of these cases below.

Case (i): In this case we can see by (4.1) that for any k, and any v,

$$T^{k,d^{-k}}(v^{-k}) - \delta \leq \bar{p}^{(1,d^{-k})} - \bar{p}^{(0,d^{-k})} \leq T^{k,d^{-k}}(v^{-k}) + \delta,$$

[7]AS implies that the image of any $T^{k,d^{-k}}$ function be strictly positive and finite.

whenever $0 < \delta < \epsilon^k(T^{k,d^{-k}}(v^{-k}), v^{-k})$.[8] This in turn implies that (**c**) $\bar{p}^{(1,d^{-k})} = \bar{p}^{(0,d^{-k})} + T^{k,d^{-k}}(v^{-k})$. This implies that for any k and any d^{-k}, the $T^{k,d^{-k}}(.)$ functions do not depend on the argument v^{-k}. Further, for any $l \neq k$, a change in v^l (by AS) may change value of d^{-k}, and so, it must be that the $T^{k,d^{-k}}$ values are independent of the value of d^{-k}. Thus, the property (1) of the (theorem) statement follows as a necessary condition for incentive compatibility. Further, by (**c**), for any $d \in \{0,1\}^m$,

$$d^k = 1 \implies \bar{p}^d = \bar{p}^{(0,d^{-k})} + T^k.$$

Therefore, $\bar{p}^d = \sum_{k:d^k=1} T^k + \bar{p}^{(0,0,\ldots,0)}$. Since by assumption, not being sold any object implies payment of zero price, the property (2) of the statement follows.

Case (ii): Fix any $k \neq l \in M$, any $\bar{v}^{-k} \in \mathbb{R}^{m-1}_{++}$, and any $\theta^k, \beta^k > T^k(\bar{v}^{-k})$. Now, if $T^l(\theta^k, \bar{v}^{-k-l}) > T^l(\beta^k, \bar{v}^{-k-l})$, then, by **(a)**, the buyer finds it profitable to report a false valuation $(\beta^k, \zeta^l, \bar{v}^{-k-l})$ at a true valuation $(\theta^k, \zeta^l, \bar{v}^{-k-l})$, where $T^l(\theta^k, \bar{v}^{-k-l}) > \zeta^l > T^l(\beta^k, \bar{v}^{-k-l})$. On the hand, if $T^l(\theta^k, \bar{v}^{-k-l}) < T^l(\beta^k, \bar{v}^{-k-l})$, then, by **(a)**, the buyer finds it profitable to report a false valuation $(\theta^k, \eta^l, \bar{v}^{-k-l})$ at the true valuation $(\beta^k, \eta^l, \bar{v}^{-k-l})$, where $T^l(\theta^k, \bar{v}^{-k-l}) < \eta^l < T^l(\beta^k, \bar{v}^{-k-l})$. Therefore, incentive compatibility implies that $T^l(x, \bar{v}^{-k-l})$ is constant for all x values greater than $T^k(\bar{v}^{-k})$. Since, $T^l(.)$ must not depend on $\bar{v}^l$, it must be that $T^k(.)$ does not depend on value of $\bar{v}^l$. Since, k, l and $\bar{v}^{-k}$ were arbitrarily chosen, $T^k(.)$ must be a constant function. Therefore, there exist positive real numbers $\{T^k\}_{k \in M}$ such that property (1) of the statement follows. Further, like in the previous case, for any k, and any v,

$$T^k - \delta \leq \bar{p}^{(1,d^{-k})} - \bar{p}^{(0,d^{-k})} \leq T^k + \delta, \quad \forall\, \delta \in (0, \epsilon^k(T^k, v^{-k})),$$

and so, like (**c**) in previous case (i), we get that $\bar{p}^{(1,d^{-k})} = \bar{p}^{(0,d^{-k})} + T^k$. Therefore, as in case (i), property (2) of the statement follows. $\square$

Proof of sufficiency: Consider any mechanism $\mu = (d, p)$ as described in the statement of the theorem. Since $\{T^k\}$ values are positive reals, for any k and any $v^{-k} \in \mathbb{R}^{m-1}_{++}$, there exists $v_0^k < T^k < v_1^k$ such that $d(v_0^k, v^{-k}) \neq d(v_1^k v^{-k})$. Thus (d, p) satisfies AS.

Now, suppose that the buyer has a value $v \in \mathbb{R}^m_{++}$. For any possible misreport v', if $d(v') = d(v)$, then by construction, $p(v') = p(v)$, and so, there can be no violation of incentive

[8] Recall that this $\epsilon^k(.)$ value has been defined earlier.

compatibility. Further, if $d(v') \neq d(v)$, then,

$$p(v') - p(v) = \sum_{k:d^k(v')=1} T^k - \sum_{k:d^k(v)=1} T^k = \sum_{k:d^k(v')-d^k(v)=1} T^k - \sum_{k:d^k(v')-d^k(v)=-1} T^k$$

Note that for any $k \in M$, if $d^k(v') - d^k(v) = 1$, then $v^k \leq T^k \leq v'^k$, and if $d^k(v') - d^k(v) = -1$, then $v'^k \leq T^k \leq v^k$ (by construction). So,

$$u((d(v'), p(v')); v) - u((d(v), p(v)); v) = \sum_{k:d^k(v')-d^k(v)=1} (v^k - T^k) - \sum_{k:d^k(v')-d^k(v)=-1} (v^k - T^k) \leq 0,$$

and hence, again, there can be no violation of incentive compatibility. Thus, the result follows.

□

4.2. **Multiple buyer case.** With multiple buyers, private information in our model becomes an $n \times m$ matrix. We show below how the Theorem 1 can be extended to this general setting.

Theorem 2. *A mechanism $\mu \in \Gamma_{M,N}$ if and only if for any $i \in N$, there exist functions $\{T_i^k : \Pi_{j \neq i} V_j \longrightarrow \mathbb{R}_{++}\}_{k \in M}$ such that for all $v \in \mathcal{V}$ and all $k \in M$:*

(1) $d_i^k(v) = \begin{cases} 1 & \text{if } v_i^k > T_i^k(v_{-i}) \\ 0 & \text{if } v_i^k < T_i^k(v_{-i}) \end{cases}$ *, and*

(2) $p_i(v) = \sum_{k:d^k(v)=1} T_i^k(v_{-i})$.

Proof of necessity: Fix any buyer i and any v_{-i}. As argued in proof of Theorem 1, we can obtain an analogue of (4.1) where for all $v_i, v_i' \in V_i$,

$$v_i^T[d_i(v_i', v_{-i}) - d_i(v_i, v_{-i})] \leq p_i(v_i', v_{-i}) - p_i(v_i, v_{-i}) \leq v_i'^T[d_i(v_i', v_{-i}) - d_i(v_i, v_{-i})]. \tag{4.2}$$

Therefore, as in case of Theorem 1, $d^i(v_i, v_{-i}) = d^i(v_i', v_{-i}) \implies p_i(v_i', v_{-i}) = p_i(v_i, v_{-i})$. Similarly, like in Theorem 1, for any $k \in M$, whenever $v_i^{-k} = v'^{-k}_i$, we can infer that

$$d_i^k(v) = \begin{cases} 1 & \text{if } v_i^k > T_i^{k,d_i^{-k},d_{-i}^{-k}}(v_i^{-k}, v_{-i}) \\ 0 & \text{if } v_i^k < T_i^{k,d_i^{-k},d_{-i}^{-k}}(v_i^{-k}, v_{-i}) \end{cases} \quad \text{for all } v \in \mathcal{V}.$$

Again, as argued in Theorem 1, we can infer that

$$T_i^{k,d_i^{-k},d_{-i}^{-k}}(v_i^{-k}, v_{-i}) \equiv T_i^{k,d_{-i}^{-k}}(v_{-i}).$$

We can further infer from Theorem 1 that the threshold for any object k, must not depend on the valuations in v_i^{-k}. This implies that this threshold would not depend on decision for any object $l \neq k$ of any other buyer, as (by AS) it can always be altered by changing v_i^l. Thus, we get that

$$T_i^{k,d_{-i}^{-k}}(v_{-i}) \equiv T_i^k(v_{-i})$$

Further, by AS, $T_i^k(v_{-i}) \in \mathbb{R}_{++}$. Thus, arguing as in case (ii) of Theorem 1, the result follows.

□

Proof of sufficiency: Consider any mechanism (d,p) as described in the statement of the Theorem 2. Fix any buyer i, and any v_{-i}. Since $\{T_i^k(v_{-i})\}_{k\in M}$ values are positive reals, and so, for any k, there exists $\underline{v}_i^k < T_i^k(v_{-i}) < \bar{v}_i^k$ such that $d((\underline{v}_i^k, v_i^{-k}), v_{-i}) \neq d((\bar{v}_i^k, v_i^{-k}), v_{-i})$. Thus (d,p) satisfies AS.

Now, consider a valuation profiles $v_i, v_i' \in V_i$. Suppose, that v_i is i's true profile, while v_i' is a misreport. If $d_i(v_i, v_{-i}) = d_i(v_i', v_{-i})$, then by construction, $p_i(v_i, v_{-i}) = p_i(v_i', v_{-i})$, and so, there can be no violation of SP. If $d_i(v_i, v_{-i}) \neq d_i(v_i', v_{-i})$, then by definition,

$$\begin{aligned}
p_i(v_i', v_{-i}) - p_i(v_i, v_{-i}) &= \sum_{k:d_i^k(v_i', v_{-i})=1} T_i^k(v_{-i}) - \sum_{k:d_i^k(v_i, v_{-i})=1} T_i^k(v_{-i}) \\
&= \sum_{k:d_i^k(v_i', v_{-i})-d_i^k(v_i,v_{-i})=1} T_i^k(v_{-i}) - \sum_{k:d_i^k(v_i', v_{-i})-d_i^k(v_i,v_{-i})=-1} T_i^k(v_{-i})
\end{aligned}$$

Note that for any $k \in M$, by construction, $d_i^k(v_i', v_{-i}) - d_i^k(v_i, v_{-i}) = 1 \implies v_i^k \leq T_i^k(v_{-i}) \leq {v'}_i^k$, while $d_i^k(v_i', v_{-i}) - d_i^k(v_i, v_{-i}) = -1 \implies v_i^k \geq T_i^k(v_{-i}) \geq {v'}_i^k$. So,

$$\begin{aligned}
&u((d(v_i', v_{-i}), p(v_i', v_{-i})); v_i) - u((d(v_i, v_{-i}), p(v_i, v_{-i})); v_i) \\
&= \sum_{k:d_i^k(v_i', v_{-i})-d_i^k(v_i,v_{-i})=1} (v_i^k - T^k(v_{-i})) \quad - \sum_{k:d_i^k(v_i', v_{-i})-d_i^k(v_i,v_{-i})=-1} (v_i^k - T^k(v_{-i})) \leq 0,
\end{aligned}$$

and hence, it follows that (d,p) does satisfies SP. Hence, the result follows. □

Remark 1. *Note that Theorem 2 allows arbitrary allotment decision for an object k, at any profile v where: (i) $v_i^k \leq T_i^k(v_{-i})$ for all $i \in N$, and (ii) there exists a subset $S^v \subseteq N$ such that for all $j \in S^v$, $v_j^k = T_j^k(v_{-j})$. Without loss of generality, henceforth we assume that at any such profile v, the object k would be sold to the maximal agent in S^v according to the linear order $1 \succ 2 \succ \ldots\ldots \succ n$.*

Theorems 1 and 2 show that each mechanism in $\Gamma_{M,N}$ is exclusively determined by the threshold functions $\{T_i^k(.)\}_{\substack{i\in N\\ k\in M}}$. This allows the class of mechanisms $\Gamma_{M,N}$ to be intractably large. Therefore, to obtain sharper results, we focus on a class of well-behaved mechanisms defined below. We first define the following metric in any space of matrices with $p \in \mathbb{N}$ rows and $q \in \mathbb{N}$ columns. For any two such matrices $A := (a_{tl})$ and $B := (b_{tl})$, we define the distance between the two as

$$||A - B|| := \sqrt{\sum_{1 \le t \le p} \sum_{1 \le l \le q} (a_{tl} - b_{tl})^2}.$$

Further, a sequence of such matrices A^j is defined to converge to limit matrix A if and only if $(||A^j - A||)_j \to A$.

Definition 4. A mechanism μ is said to be *continuous* if for any $\zeta \in \{0, 1\}$, any $i \in N$, any $k \in M$, and any sequence of profiles $\{v^l\}$ that converges to $\tilde{v}$; whenever $d_i^k(v^l) = \zeta$ for all l,

$$d_i^k(\tilde{v}) \neq \zeta \implies \Big[u(d_i(\tilde{v}), p_i(\tilde{v}); \tilde{v}_i) = u((\zeta, d_i^{-k}(\tilde{v})), p_i(\tilde{v}); \tilde{v}_i)\Big]$$

Let $\bar{\Gamma}$ be the class of continuous mechanisms in $\Gamma_{M,N}$.

We use the above-mentioned notion of distance to define a concept of concept of continuity of mechanisms in our setting. A mechanism is *continuous* if it satisfies the property that: whenever the allotment decision of a buyer i is not preserved in limit, the transfer assigned to i at the limit profile is such that she is indifferent between getting and not getting the object. This rules out mechanisms where allotment decisions change in peculiar manner.[9]

As argued in Thomson [30], the notion of continuity of a mechanism has appealing strategic as well as ethical characteristics. In highlighting its strategic value, Thomson [30] states that "*...a discontinuous rule is likely to be manipulable in undetectable ways*"; while in specifying its ethical value, he posits that continuity rules out unfair situations where small changes in underlying preferences (which may arise due to involuntary inaccurate reporting) result in stark changes of buyers' welfare.

The following proposition presents a remarkable result about continuous mechanisms in $\Gamma_{M,N}$ that sell all objects at all profiles. It shows that any such mechanism is equivalent to a *simple* mechanism that sells each object through a separate single object auction. To present this result, we first formally define the class of simple mechanisms $\Gamma^S_{M,N}$.

Definition 5. A mechanism $\mu \in \Gamma_{M,N}$ is said to be *simple* if for all $i \in N$, $k \in M$, and $v \in \mathcal{V}$;

$$T_i^k(v_{-i}) = f_i(v_{-i}^k),$$

where $f_i(.)$ is a real valued function defined on $\mathbb{R}^{n-1}_{++}$. We denote the subset of such simple mechanisms by $\Gamma^S_{m,n}$.

[9]To see an example of such peculiar behaviour, consider a mechanism in $\Gamma_{\{1,2\},\{1,2,3\}}$ that satisfies AS and SP but not continuity:

$$T_i^k(v_{-i}) = \max_{j \neq i} v_j^k + \min_{j \neq i} v_j^k, \text{ for any } i \neq j, k, \text{ and } v.$$

Remark 2. *Note that using a simple mechanism $\mu \in \Gamma^S_{m,n}$ to sell m objects is equivalent to using m separate single object auctions among the set of buyers N, one for each different object. Further, in the auction for any object $k \in M$, a buyer i pays a positive price $T^k_i(.)$ if and only if she wins k.*

Theorem 3. If a mechanism $\mu \in \bar{\Gamma}_{M,N}$, then $\mu \in \Gamma^S_{M,N}$.

Proof: Fix any mechanism $\mu \in \bar{\Gamma}_{M,N}$, any buyer $i \in N$ and any object $k \in M$. The proof is accomplished using the following two steps.

Step 1: In this step, we show that $T^k_i(.)$ is a continuous function.

Suppose not. Then there exists a sequence $(v^n_{-i})_n \to \bar{v}_{-i}$ such that $(T^k_i(v^n_{-i}))_n$ does not converge to $T_i(\bar{v}_{-i})$. Therefore, there exists an $\varepsilon > 0$, and a subsequence $(T^k_i(v^{n^l}_{-i}))_l$ such that for all $l \in \mathbb{N}$, $|T^k_i(v^{n^l}_{-i}) - T^k_i(\bar{v}_{-i})| \geq \varepsilon$. Without loss of generality, we assume that for all $l \geq 1$, **(a)** $T^k_i(v^{n^l}_{-i}) \geq T^k_i(\bar{v}_{-i}) + \varepsilon$.[10] Now, define a v_i such that $v^t_i = T^t_i(\bar{v}_{-i}) + \frac{\varepsilon}{2}$ for all $t \in M$, and consider the sequence of profiles $\Big((v_i, v^{n^l}_{-i})\Big)_l$. By supposition $\Big((v_i, v^{n^l}_{-i})\Big)_l \to (v_i, \bar{v}_{-i})$, and by **(a)**, for all $l \geq 1$, $v^k_i < T^k_i(\bar{v}_{-i}) + \varepsilon \leq T^k_i(v^{n^l}_{-i})$. Therefore, by Theorem 2, $d^k_i(v_i, v^{n^l}_{-i}) = 0$ for all l, while $d^k_i(v_i, \bar{v}_{-i}) = 1$. Hence, continuity of μ implies that $v^k_i - T^k_i(\bar{v}_{-i}) = 0 \implies \varepsilon = 0$, which is a contradiction. Thus, $(T^k_i(v^n_{-i}))_n \to T_i(\bar{v}_{-i})$, and so, we can infer that $T^k_i(.)$ functions are continuous.

Step 2: In this step, we show that for any $i \in N$, any $v \in \mathcal{V}$ and any $k \in M$: **(I)** $\big[T^k_i(v_{-i}) < v^k_i\big] \implies \Big[\forall\, j \neq i, T^k_j(v_{-j}) > v^k_j\Big]$, and **(II)** $\big[T^k_i(v_{-i}) = v^k_i\big] \implies \Big[\forall\, j \neq i, T^k_j(v_{-j}) \geq v^k_j\Big]$.

Suppose **(I)** is not true. That is, suppose that there exist $j \neq i$, v and k such that $T^k_i(v_{-i}) < v^k_i$ and $T^k_j(v_{-j}) = v^k_j$. By Step 1, there exists a $\delta > 0$ such that $||z - v_{-i}|| < \delta \implies v^k_i > T^k_i(z)$. Now consider the profile $\hat{v}$ where $\hat{v}_{-j} = v_{-j}$, $\hat{v}^k_j = v^k_j + \frac{\delta}{2}$, and $\hat{v}^t_j = v^t_j$, for all $t \in M \setminus \{k\}$. By construction, $\hat{v}^k_i > T^k_i(\hat{v}_{-i})$ and $\hat{v}^k_j > T^k_j(\hat{v}_{-j})$, implying that $d^k_i(\hat{v}) = d^k_j(\hat{v}) = 1$, which is a contradiction. So **(I)** is true, and arguing in the same manner, we can show that **(II)** is true.

[10] To be formally exact, there must exist a further subsequence $\Big(T^k_i(v^{n^{l^t}}_{-i})\Big)_t$ such that either $\Big[T^k_i(v^{n^{l^t}}_{-i}) \geq T^k_i(\bar{v}_{-i}) + \varepsilon, \forall\, t \geq 1\Big]$ or $\Big[T^k_i(v^{n^{l^t}}_{-i}) \leq T^k_i(\bar{v}_{-i}) - \varepsilon, \forall\, t \geq 1\Big]$. In the proof, we work with the former possibility (with reduced notation for simplicity) to arrive at a contradiction below. It can easily be shown that a contradiction arises using a similar logic for the latter possibility too.

Now, fix any $j \neq i$, and define a function $G_j^k(v) : \mathcal{V} \mapsto \mathbb{R}$ such that $\forall\ v \in \mathcal{V}$, $G_j^k(v) := v_j^k - T_j^k(v_{-j})$. Note that by **(I)** and AS,[11]

$$(\mathbf{b}) \quad G_j^k(v) = \begin{cases} \text{negative} & \text{if } v_i^k > T_i^k(v_{-i}) \\ \text{ambiguous} & \text{if } v_i^k \leq T_i^k(v_{-i}). \end{cases}$$

Thus, for any profile v, by construction, $G^k(v)$ cannot depend on v_j^{-k}, while by **(b)**, it depends on v_{-i}. Therefore, condition **(b)** can hold true only if $T_i^k(v_{-i})$ does not depend on v_j^{-k}. And since i, j, k, and v were chosen arbitrarily, we can infer that $T_i^k(v_{-i}) = f_i(v_1^k, \ldots, v_{i-1}^k, v_{i+1}^k, \ldots, v_n^k), \forall i, k, v$. Hence, the result follows. □

Theorem 3 shows that all continuous mechanisms in $\Gamma_{M,N}$ can be implemented via suitably chosen m separate single object auctions among the n buyers. This result highlights an additional advantage of our continuity condition, as it greatly simplifies implementation of complicated heterogeneous object allocations where buyers may demand multiple objects (like spectrum auctions). This simplicity is crucial for practical implementation of any mechanism, and its relevance is noted in the following quote by Nobel laureate Robert Wilson as noted in Milgrom [17]:[12]

> *"... Wilson doctrine, which holds that practical mechanisms should be simple and designed without assuming that the designer has very precise knowledge about the economic environment in which the mechanism will operate."*

Additionally, as stated in the following corollary, Theorem 3 also has an important technical implication for continuous mechanisms in $\Gamma_{M,N}$.

Corollary 1. A mechanism $\mu \in \bar{\Gamma}_{M,N}$ if and only if for all $i \in N$, $k \in M$, and $v \in \mathcal{V}$,

$$T_i^k(v_{-i}) = f_i(v_1^k, \ldots, v_{i-1}^k, v_{i+1}^k, \ldots, v_n^k),$$

where $f_i(.)$ is continuous real valued function.

Proof: The necessity result follows from Theorem 3, and the proof of sufficiency is easy to check. □

Thus, Corollary 1 establishes the continuous nature of the threshold functions associated with any mechanism in $\bar{\Gamma}_{M,N}$.

[11] AS eliminates the possibility that $G_j^k(v)$ is always negative irrespective of whether $v_i^k > T_i^k(v_{-i})$ or not.
[12] See footnote number 22 in page 23 of Milgrom [17].

We now proceed to present the first main result of this paper, which states that the basic ethical notion of anonymity, coupled with the continuity restriction, generates decision efficiency for strategyproof and agent sovereign mechanisms that sell all objects at all profiles. This idea of efficiency is formally defined below.

Definition 6. *A mechanism μ^e is efficient (EFF) if for all $v \in \mathcal{V}$,*

$$\sum_{i \in N} d_i^{\mu^e} v_i = \max_{\hat{d} \in \mathcal{D}} \sum_{i \in N} \hat{d}_i v_i.$$

Thus, an efficient mechanism must choose the decision matrix in $\mathcal{D}$ that maximizes social utility at possible valuation profiles. More specifically, given a valuation profile v (that is a matrix of (v_i^k) with n rows and m columns), an efficient mechanism μ^e replaces any of the largest values in each column by 1, and all other values by 0; to generate the resultant decision matrix $d^{\mu^e}(v)$. In other words, an efficient mechanism gives each object k to the bidder who bids the maximum for it. Since bidders pay no price if they do not win any object, by the famous characterization result of Holmström [13], any efficient mechanism in $\Gamma_{M,N}$ must be the following *Pivotal* mechanism.

Definition 7. *A mechanism μ^P is said to be the Pivotal mechanism if for all $i \in N$ and all $v \in \mathcal{V}$,*

$$\sum_{t \in N} d_t^{\mu^P} v_t = \max_{\hat{d} \in \mathcal{D}} \sum_{t \in N} \hat{d}_t v_t \quad \textit{and} \quad p_i^{\mu^P}(v) = \max_{\hat{d} \in \mathcal{D}} \sum_{\substack{t \in N \\ t \neq i}} \hat{d}_t v_t - \sum_{\substack{t \in N \\ t \neq i}} d_t^{\mu^P} v_t.$$

The following result posits that equity and efficiency are closely related in our model, because any anonymous and continuous mechanism in $\Gamma_{M,N}$ that always sells all available objects, must be decision efficient. This result is formally presented below.

Theorem 4. Consider any mechanism $\mu \in \bar{\Gamma}_{M,N}$ that sells all objects at all profiles.[13] Then, μ satisfies AN if and only if it satisfies EFF.

Proof: The proof of necessity is presented in the Appendix. To see the proof of sufficiency, recall that by Holmström [13], any efficient mechanism in our setting (where buyers pay only if they win an object, and the domain of valuations is smoothly connected), must be the Pivotal mechanism μ^P. Therefore, for all $i \in N$ and all $v \in \mathcal{V}$:

$$\sum_{t \in N} d_t^{\mu^P} v_t = \max_{\hat{d} \in \mathcal{D}} \sum_{t \in N} \hat{d}_t v_t \quad \text{and} \quad p_i^{\mu^P}(v) = \sum_{\substack{t \in N \\ t \neq i}} d_t^{\mu^P} v_t - \max_{\hat{d} \in \mathcal{D}} \sum_{\substack{t \in N \\ t \neq i}} \hat{d}_t v_t.$$

[13]KMS [14] refer to such mechanisms as 'no-wastage' mechanisms.

Therefore, for all i and all v,

$$u(\mu_i^P(v); v_i) = \sum_{\substack{k \in M \\ d_i^k(v)=1}} \left[v_i^k - \max_{j \neq i} v_j^k \right],$$

that is, the Pivotal mechanism is equivalent to executing m different *"Second Price Auction"*s - one for each object to be sold.

Note that for any bijection $\pi : N \mapsto N$ and any $i \in N, k \in M$; $v_i = (\pi v)_{\pi i}$, $\max_{j \neq i} v_j^k = \max_{j \neq \pi i} (\pi v)_j^k$, and so, $u(\mu_i^P(v); v_i) = u(\mu_{\pi i}^P(\pi v); (\pi v)_{\pi i})$. Therefore, it is easy to see that μ^P satisfies anonymity. Further, fix any $i \in N$, any $k \in M$, and (without loss of generality) consider any sequence $({}^l v)_l \to \tilde{v}$ such that $d_i^{k\mu^P}({}^l v) = 1$ with $d_i^{k\mu^P}(\tilde{v}) = 0$. Therefore, $(\max_{j \neq i} {}^l v_j^k)_l$ converges to $\max_{j \neq i} \tilde{v}_j^k$, and so,

$$d_i^{k\mu^P}({}^l v) = 1 \text{ for all } l \implies {}^l v_i^k \geq \max_{j \neq i} {}^l v_j^k \text{ for all } l \implies \tilde{v}_i^k \geq \max_{j \neq i} \tilde{v}_j^k.$$

Therefore, $d_i^{k\mu^P}(\tilde{v}) = 0$, which implies $\tilde{v}_i^k = \max_{j \neq i} \tilde{v}_j^k$, which in turn implies that $u(d_i^{\mu^P}(\tilde{v}), p_i^{\mu^P}(\tilde{v}); \tilde{v}_i) = u((1, d_i^{-k\mu^P}(\tilde{v})), p_i^{\mu^P}(\tilde{v}); \tilde{v}_i)$. Thus, μ^p satisfies continuity. Finally, it is easy to see that Pivotal mechanism satisfies strategyproofness and agent sovereignty. Thus, $\mu^P \in \bar{\Gamma}_{M,N}$. □

Theorem 4 establishes that any anonymous mechanism in $\bar{\Gamma}_{M,N}$ must be efficient. Therefore, as argued earlier, from Holmström [13] it follows that the only mechanism in $\bar{\Gamma}_{M,N}$ that is anonymous in our setting, is the Pivotal mechanism. This idea is formalized in the corollary below.

Corollary 2. If a mechanism $\mu \in \bar{\Gamma}_{M,N}$ that sells all objects at all profiles satisfies AN, then $\mu = \mu^P$.

Proof: Since buyers pay only if they win an object in our setting, by Holmström [13], Theorem 4 implies that the only anonymous mechanism in $\bar{\Gamma}_{M,N}$ is the Pivotal mechanism. □

Remark 3. *As noted in proof of Theorem 4, there may be several different ways of executing such a Pivotal mechanism, each with a separate algorithm to generate an efficient object allocation. One simple and elegant way of implementing Pivotal mechanism is to conduct a separate simultaneous sealed bid second price auction for each object. As noted in Mueller* [23]*, Government of New Zealand used this method to sell cellular management right tenders in 1990.*

They were advised this manner of spectrum allocation by the reputed British-American consultancy firm National Economic Research Associates (NERA). Our paper, therefore, provides an axiomatic foundation to this procedural advice.[14]

Now, Theorem 4 focusses on mechanisms that allot all objects at all profiles like KMS [14]. Yet, one could think of mechanisms that, a priori, allow a subset of objects to remain unsold. The most common of such mechanisms would be the reserve price mechanisms, where objects are not sold unless bids received are high enough. The next theorem characterizes these mechanisms using the following *non-bossiness* property that requires decision functions to be reasonably well behaved, while imposing no restrictions on the transfer function.[16]

Definition 8. *A mechanism $\mu = (d, \tau)$ is non-bossy in decision if for any $i \in N$, and any $v, \bar{v} \in \mathcal{V}$ such that $v_i \neq \bar{v}_i$ and $v_{-i} = \bar{v}_{-i}$;*

$$d_i(v) = d_i(\bar{v}) \Longrightarrow \forall j \neq i, d_j(v) = d_j(\bar{v})$$

Let $\hat{\Gamma}_{M,N}$ be the set of all such mechanisms in $\bar{\Gamma}_{M,N}$.

Thus, a mechanism belongs to $\hat{\Gamma}_{M,N}$ if and only if it has a well behaved decision function where no buyer can unilaterally change her reported valuation in such a way that her object allotment remains unchanged, but some other buyer's object allotment changes. As discussed in Thomson [30], apart from manifesting a reasonable fairness notion, this property embodies a strategic barrier to collusive practices where buyers form groups to misreport in a manner that changes the allotment decision to benefit any one member of the group while not making any other member worse off.

We show below that any continuous strategyproof and agent sovereign mechanism satisfying anonymity in $\hat{\Gamma}_{M,N}$ must use object specific reserve prices.

Theorem 5. Consider any mechanism $\mu \in \hat{\Gamma}_{M,N}$. The mechanism μ satisfies AN if and only if there exists a set of non-negative real numbers $\{r^k\}_{k \in M}$ such that for each object k:

[14] It must be mentioned here that these auctions generated political controversy on account of the winners at second price auctions paying far lower second highest bid as price, and this led to substitution of this second price auction with first price auction for the future rounds of spectrum allocations. However, given that this was the first ever spectrum allocation exercise conducted by New Zealand, Mueller [23] argues that it is likely that this divergence in bids was generated by "*thinness of the New Zealand market and the large quantity of spectrum released at once*".[15] Further, Mueller [23] notes that the primary motive of spectrum allocation in New Zealand in 1990, was "*the creation of a market regime rather than revenue generation*". Yet auctions of these cellular tenders raised \$36.358 million out of the total revenue of \$45.6 million from sale of radio spectrum. Finally, Crandall [6] argues that prices realized in the aforementioned New Zealand spectrum auction are "*very similar*" to those obtained in U.S. Personal Communications Services (PCS) auctions after "*adjusting for differences in demographics.*"

[16] Similar notions of non-bossiness have been used by Svensson [29], Goswami, Mitra and Sen [10], Mishra and Quadir [20] etc.

- k is sold to the highest bidder who bids a value for k that is at least as great as r^k, or else it remains unsold,
- the winner of k pays the greater of the two amounts, r^k and the second highest bid for k, and
- $r^k = \inf\{x > 0 : \exists\, v \in \mathcal{V} \ni v^k = x\mathbf{1}^n \text{ and } d^k(v) \neq \mathbf{0}^n\}$.

Proof: The proof of sufficiency is easy to check. We present the proof of necessity below. Fix any $\mu \in \hat{\Gamma}_{M,N}$, any $i \in N$, and any $k \in M$. By Theorem 3, $\mu \in \bar{\Gamma}_{M,N} \implies \mu \in \Gamma^S_{M,N}$, and so, by Corollary 1, $T^k_i(v_{-i}) = f_i(v^k_{-i})$ for any $v \in \mathcal{V}$ where $f_i(.)$ is a functional. Now, suppose there exists an $x > 0$ and a profile v^0 such that: (i) $v^{0k} = x\mathbf{1}^n$, and (ii) $d^k(v^0) \neq \mathbf{0}^n$. Further, fix any $\varepsilon > 0$, and consider the profile v^ε such that $v^{\varepsilon k} = (x + \varepsilon)\mathbf{1}^n$, $v^{\varepsilon -k} = v^{0-k}$.
Now, construct a recursive sequence of profiles $\{{}^h v\}_{h=1}^{n+1}$ such that ${}^1v := {}^\varepsilon v$, for any $1 < q \leq n+1$,

$$ {}^q v^{-h} = {}^r v^{-h},\ {}^q v^h_r = x,\ {}^r v^h_{-r} = {}^q v^h_{-r} \text{ where r: =q-1.} $$

Note that ${}^{n+1}v := v^0$. Since $\mu \in \hat{\Gamma}_{M,N}$, by non-bossiness in decision, Theorem 3 and Theorem 2, any two consecutive profiles in the sequence have the same associated decision matrix. Hence, $d^k({}^h v) = \mathbf{0}^n \implies d^k({}^{h+1}v) = \mathbf{0}^n$ for all $h \leq n$, and so, $d^k(v^\varepsilon) = \mathbf{0}^n \implies d^k(v^0) = \mathbf{0}^n$, which is contradiction to the construction of v^0. Since ε was arbitrarily chosen, we can infer that: if there exists $x > 0$ satisfying properties (i) and (ii) above, then for any $y \geq x$ and any profile $v \in \mathcal{V}$,

$$ \left\{ v^k = y\mathbf{1}^n \text{ and } v^{-k} = v^{0-k} \right\} \implies d^k(v) \neq \mathbf{0}^n. $$

Since $\mu \in \bar{\Gamma}_{M,N}$; given the non-negative real number $r^k := inf\{x > 0 : \exists\ v \text{ such that } v^k = x\mathbf{1}^n \text{ and } d^k(v) \neq \mathbf{0}^n\}$, we can infer that:

$$ (\mathbf{a})\ \ z > r^k \implies \left\{ \forall\, v \text{ such that } v^k = z\mathbf{1}^n,\ d^k(v) \neq \mathbf{0}^n \right\}, $$

and

$$ (\mathbf{b})\ \ z < r^k \implies \left\{ \forall\, v \text{ such that } v^k = z\mathbf{1}^n, d^k(v) = \mathbf{0}^n \right\}. $$

Now, consider any profile v^* and suppose, without loss of generality that, $v^{*k}_1 \geq v^{*k}_2 \geq \ldots \geq v^{*k}_n$.[17] There are three possibilities: (I) $r^k > v^{*k}_1$, (II) $r^k < v^{*k}_1$, and (III) $r^k = v^{*k}_1$. We consider each of these below.

[17]Note that this preservation of generality is due to Proposition 2 which implies that for any anonymous mechanism in $\bar{\Gamma}_{M,N}$; the threshold function for each object is independent of buyer identity.

Case (I): By **(b)**, $d^k(v') = \mathbf{0}^n$ for any profile v' with $v'^k := {v^*}^k_1 \mathbf{1}^n$ and $v'^{-k} = v^{*-k}$. Construct a sequence of profiles $\{{}^t w\}_{t=1}^n$ such that ${}^1 w := v'$, and for all $2 \le t \le n$,

$$ {}^{t-1}w^{-k} = {}^t w^{-k},\ {}^t w^k_t = {v^*}^k_t, \text{ and } {}^{t-1}w^k_{-t} = {}^t w^k_{-t}. $$

By Theorem 2, $d^k_t({}^t w) = 0$ for all $t > 1$, and so, by non-bossiness of decision, Theorem 3 and Theorem 2, as before, any two consecutive profiles in the sequence have the same associated decision matrix. Hence, $d^k({}^t \bar{v}) = d^k({}^{t+1}\bar{v}) = \mathbf{0}^n$ for all $t \le n-1$. Since $d^k({}^1 w) = d^k(v') = \mathbf{0}^n$, we can infer that $d^k({}^n w) = \mathbf{0}^n$. By construction ${}^n w = v^*$, and so, we get that $(\mathbf{A})$ $d^k(v^*) = \mathbf{0}^n$.

Case (II): Consider the profile $\hat{v}$ such that $\hat{v}^{-k} = v^{*-k}$ and $\hat{v}^k = \eta \mathbf{1}^n$, where $\eta := \frac{{v^*}^k_1 + \max\{r^k, {v^*}^k_2\}}{2}$. By **(a)**, $d^k(\hat{v}) \neq \mathbf{0}^m$, and so, by Propositions 1 and 2 in Appendix, $T^k_j(\hat{v}_{-j}) = \eta$ for all $j \in N$. Therefore, by Theorem 2, for the profile $\bar{\bar{v}}$ such that $\bar{\bar{v}}^{-k} = \hat{v}^{-k}$, $\bar{\bar{v}}^k_{-1} = \hat{v}^k_{-1}$, and $\bar{\bar{v}}^k_1 = {v^*}^k_1$,

$$ d^k_1(\bar{\bar{v}}) = 1. $$

Now, consider a sequence of profiles $\{{}^h w\}_{h=1}^n$ where ${}^1 w := \bar{\bar{v}}$, and for all $2 \le h \le n$,

$$ {}^{h-1}w^{-k} = {}^h w^{-k},\ {}^h w^k_h = {v^*}^k_h, \text{ and } {}^{h-1}w^k_{-h} = {}^h w^k_{-h}. $$

As argued above, by non-bossiness, Theorem 3 and Theorem 2, $d({}^h w) = d({}^{h+1} w)$ for all $h \le n-1$, and so, $d^k_1(w^1) = 1 \implies d^k_1({}^n w) = 1$. By construction, ${}^n w = v^*$, and so, we get that $(\mathbf{B})$ $d^k_1(v^*) = 1$.

Case (III): Consider a profile $\tilde{v}$ where $\tilde{v}^k = r^k \mathbf{1}^n$, and a sequence of profiles $\{{}^l w\}_l$ such that:

- for any $l \in \mathbb{N}$, there exists a real ${}^l\theta > r^k$ such that ${}^l w^k = {}^l\theta \mathbf{1}^n$ and ${}^l w^{-k} = \tilde{v}^{-k}$, and
- $\{{}^l\theta\}_l \to r^k$.

By condition **(a)**, for all l, $d^k({}^l w) \neq \mathbf{0}^n$. And so, by Propositions 1 and 2, $T^k_i({}^l w_{-i}) = {}^l\theta$ for all $i \in N$. By Corollary 1 and the construction of the sequence $\{{}^l w\}_l$, for all i, $\{T^k_i({}^l w_{-i})\}_l \to T^k_i(\tilde{v}_{-i})$, which in turn implies that $T^k_i(\tilde{v}_{-i}) = r^k$. Therefore, by Remark 1, $d^k_1(\tilde{v}) = 1$.

Now consider the sequence of profiles $\{{}^t\bar{v}\}_{t=1}^n$, where ${}^1\bar{v} := \tilde{v}$, and for all $2 \le t \le n$,

$$ {}^{t-1}\bar{v}^{-k} = {}^t\bar{v}^{-k},\ {}^t\bar{v}^k_t = {v^*}^k_t, \text{ and } {}^{t-1}\bar{v}^k_{-t} = {}^t\bar{v}^k_{-t}. $$

Again as argued earlier, by non-bossiness of decision, Theorem 3 and Theorem 2, $d({}^{t}\bar{v}) = d({}^{t+1}\bar{v})$ for all $t \leq n-1$, and so, $d_1^k({}^{1}\tilde{v}) = 1 \Longrightarrow d_1^k({}^{n}\bar{v}) = 1$. By construction, ${}^{n}\bar{v} = v^*$, and so, we get that **(C)** $d_1^k(v^*) = 1$.

Recall that the profie v^* was chosen arbitrarily without any loss of generality. Therefore, findings **(A)**, **(B)** and **(C)** taken together imply that; for any object k, there exists a real number $r^k \geq 0$ such that

$$T_i^k(v_{-i}) = \max\{r^k, \max_{j \neq i} v_j^k\}, \forall\, i \in N, \forall\, v \in \mathcal{V}.$$

Thus, by Theorem 2, the result follows. □

Remark 4. *As noted in Remark 3, a simple and elegant manner of implementing mechanisms characterized by Theorem 5 is to hold simultaneous second price auctions with (possibly different) reserve prices for each object.*

5. Discussion

A setting of heterogeneous object allocation allows us to motivate the notions of '*complementarity*' and '*substitutability*'.[18] The former idea describes situations where buyers perceive high value of an object if they can use it in conjunction with other objects. In such cases, upon acquiring an object, a buyer is likely to be willing to pay more for other objects. The latter idea represents situations where acquiring one object would make the buyers less willing to pay for other objects at auction. Absence any of these kinds of preferences is often referred to as objects being '*independent*'.

In our model, the assumption of linear buyers' preferences fails to accommodate the possibility of complementarity, as each buyers can report only m valuation numbers. Note that such interdependence across objects in buyer preferences can truly be captured, only if combinatorial bidding is allowed where each buyer places $2^{|M|} - 1$ bids (one for each possible subset of M). However, such a combinatorial auction is difficult to execute for many reasons including computational difficulties.[19] Further, in practical applications involving country wide resource rights like spectrum auctions (as noted in Milgrom [17] in the context of the aforementioned U.S. PCS auctions) '*large geographic scope*' of licenses dilutes '*the force of the argument*' that there may be complementarities among objects.[20]

Our model, however, is capable of accommodating substitutability in buyer preferences in the following manner. Buyers may identify the target objects; and bid arbitrarily small valuations

[18]See page 8 in Milgrom [17].
[19]See page 13 in the Ministry of Business, Innovation & Employment, Government of New Zealand report [25].
[20]See page 12 in Milgrom [17].

for non-target objects. Such a behaviour is observed widely enough to be known as '*parking*' (noted in the Ministry of Business, Innovation & Employment, Government of New Zealand report [25]).[21]

6. Conclusion

In our model of heterogeneous object allocation, we present a characterization of the class of strategyproof and agent sovereign mechanisms. We show that equity and efficiency are closely related, as any anonymous, agent sovereign, continuous and strategyproof mechanism selling all objects must be a decision efficient one. Consequently, by Holmström [13], the only such mechanism in our setting is the Pivotal mechanism. One obvious method of implementing Pivotal mechanism is to conduct separate simultaneous sealed bid second price auctions, as was done by New Zealand government in allocating cellular management rights tenders in 1990. Thus, our results provide an axiomatic justification to this method of allocating heterogeneous objects.

We also consider mechanisms that do not sell all objects at all profiles. We show that any such mechanism satisfies the aforementioned properties and a non-bossiness property, if and only if it employs object specific reserve prices, and sells each object to the highest bidder for that object who bids no less than the respective reserve price.

7. Appendix

7.1. **Proof of Necessity of Theorem 4.** To establish this result, we need to prove the following propositions. Recall that, for any $v \in \mathcal{V}$, and any $i \in N$, $O_i(v) := \{k \in M | d_i^k(v) = 1\}$ is the set of objects sold to buyer i at profile v.

Proposition 1. If a mechanism $\mu \in \bar{\Gamma}_{M,N}$ satisfies AN, then for any $x > 0$, $k \in M$ and $v \in \mathcal{V}$ such that $v^k = x\mathbf{1}^n$;

$$k \in O_i(v) \implies T_i^k(v_{-i}) = x.$$

Proof: Fix any positive real number x. Recall that, by Theorem 3, $\mu \in \bar{\Gamma}_{M,N} \implies \mu \in \Gamma^S_{M,N}$, and so, $T_i^k(v_{-i}) = f_i(v_{-i}^k)$ for any $v \in \mathcal{V}$, any $i \in N$, and any $k \in M$. We use this result to accomplish the proof for the following two cases.

Case 1: $m < n$.

Consider a profile $\bar{v}$ such that $v_t^l = x$ for all $t \in N$ and all $l \in M$. Now, as $m < n$, there exists a $j' \in N$, such that $O_{j'}(\bar{v}) = \emptyset$, and so, $u(\mu_{j'}(\bar{v}); \bar{v}_{j'}) = 0$. Since μ satisfies AN, $u(\mu_{j'}(\bar{v}); \bar{v}_{j'}) =$

[21]See footnote 7 in page 8 in Government of New Zealand report [25].

$u(\mu_t(\bar{v}); \bar{v}_t) = 0$ for all $t \in N$. Therefore, by Theorem 2, it follows that for any buyer t, and any object l: $d_t^l(\bar{v}) = 1 \implies x = T_t^l(\bar{v}_{-t})$. By Theorem 3, the result follows.

Case 2: $m \geq n$

Consider the same profile $\bar{v}$ such that for $\bar{v}_t^l = x, \forall\, t \in N, \forall\, l \in M$. Note that, if there exists any buyer j such that $u(\mu_j(\bar{v}); \bar{v}_j) = 0$; then, as argued in the previous case, the result follows.

Now, suppose that there does not exist any buyer j with $u(\mu_j(\bar{v}); \bar{v}_j) = 0$, that is, $u(\mu_j(\bar{v}); \bar{v}_j) > 0$ for all j.[22] Then, for each $k \in M$; if k is sold at profile matrix $\bar{v}$, there exists an $a_{k,\bar{v}} \in N$ such that $d^k_{a_{k,\bar{v}}}(\bar{v}) = 1$, and $\cup_{k \in M}\{a_{k,\bar{v}}\} = N$. Further, $\bar{v}^k_{a_{k,\bar{v}}} = x \geq T^k_{a_{k,\bar{v}}}(\bar{v}_{-a_{k,\bar{v}}})$ for each sold object k, and $O_t(\bar{v}) \neq \emptyset$ for each buyer t. Now, fix any $i \in N$ and any $\bar{k} \in O_i(\bar{v})$ such that $x > T_i^{\bar{k}}(\bar{v}_{-i})$.[23] So, by Corollary 1, there exists an $\bar{\varepsilon} > 0$ such that:

$$\textbf{(a)}\ \ ||v_{-i} - \bar{v}_{-i}|| < \bar{\varepsilon} \implies T_i^{\bar{k}}(v_{-i}) < x.$$

Consider the profile $\tilde{v}$ where: (i) $\tilde{v}_t = \bar{v}_t$ for all $t \neq i$, (ii) $\tilde{v}_i^l = \bar{v}_i^l = x$ for all $l \neq \bar{k}$, and (iii) $\tilde{v}_i^{\bar{k}} = x + \eta$ where $\eta \in (0, \frac{\bar{\varepsilon}}{2})$. From Theorem 2 and Theorem 3 it follows that: $d_i^{\bar{k}}(\tilde{v}) = d_i^{\bar{k}}(\bar{v}) = 1$, and $d_t^l(\tilde{v}) = d_t^l(\bar{v})$ for all buyers $t \neq i$ and all objects $l \neq \bar{k}$. Therefore, $u(\mu_i(\tilde{v}); \tilde{v}_i) = \mathbf{x} + \eta + \sum_{l \in O_i(\tilde{v})}[x - T_i^l(\mathbf{x})]$, where $\mathbf{x} := (x, x, \ldots, x) \in \mathbb{R}^{n-1}_{++}$. Now, fix any buyer $h \neq i$, and consider a bijection $\pi : N \mapsto N$ such that $\pi i = h$, $\pi h = i$, and for all $t \neq i, h$, $\pi t = t$. By AN, $u(\mu_{\pi i}(\pi\tilde{v}); (\pi\tilde{v})_{\pi i}) = u(\mu_i(\tilde{v}); \tilde{v}_i)$, and so, we get that:

$$\textbf{(b)}\ \ d_h^{\bar{k}}(\pi\tilde{v})\left\{(x+\eta) - T_h^{\bar{k}}(\mathbf{x})\right\} + \sum_{\substack{l \in O_h(\pi\tilde{v}) \\ l \neq \bar{k}}} \left\{x - T_h^l(\mathbf{x})\right\} = \mathbf{x} + \eta + \sum_{l \in O_i(\tilde{v})} \left\{x - T_i^l(\mathbf{x})\right\}.$$

Note that **(b)** must hold true for all $\eta \in (0, \frac{\bar{\varepsilon}}{2})$, implying that $d_h^{\bar{k}}(\pi\tilde{v}) = 1$. Therefore, $d_i^{\bar{k}}(\pi\tilde{v}) = 0$, which implies that $(\pi\tilde{v})_i = x \leq T_i^{\bar{k}}((\pi\tilde{v})_{-i})$. By construction, $||\pi\tilde{v}_{-i} - \bar{v}_{-i}|| < \bar{\varepsilon}$, and so, we get a contradiction to **(a)**. Hence, by Theorem 3, the result follows. $\square$

Proposition 2. If a mechanism $\mu \in \bar{\Gamma}_{M,N}$ satisfies AN, then for any $i, j \in N$, any $k \in M$, and any $v, \tilde{v} \in \mathcal{V}$,

$$v^k_{-i} = \tilde{v}^k_{-j} \Longrightarrow T_i^k(v_{-i}) = T_j^k(\tilde{v}_{-j}).$$

Proof: Suppose not. Then, there exists a mechanism $\mu \in \bar{\Gamma}_{M,N}$, a buyer i, an object k, and a vector $z \in \mathbb{R}^{m-1}_{++}$, and profiles $v, \tilde{v}$ such that $v^k_{-i} = \tilde{v}^k_{-i} = z$ and $T_i^k(v_{-i}) < T_j^k(\tilde{v}_{-i})$. By Theorem

[22]By Theorem 2, utility from any strategyproof mechanism is non-negative.

[23]By supposition, $u(\mu_i(\bar{v}); \bar{v}_i) > 0$, and so, such a $\bar{k} \in O_i(\bar{v})$ is well defined. Further, if i and $\bar{k}$ are not well defined, then the result follows trivially.

3, $T_i^k(v_{-i}) = f_i(z)$ and $T_j^k(\tilde{v}_{-i}) = f_j(z)$. Therefore, for economy of notation, henceforth in this subsection, we denote $T_i^k(v_{-i})$ and $T_j^k(\tilde{v}_{-i})$ as $T_i^k(z)$ and $T_j^k(z)$, respectively.
Now, fix any $\beta \in \left(T_i^k(z), T_j^k(z)\right)$, and consider the profile $\hat{v}$ such that for all buyers t, and all objects $l \neq k$, $\hat{v}_t^l = \alpha > 0$, $\hat{v}_{-i}^k = z$, and $\hat{v}_i^k = \beta$. By Theorem 2, $d_i^k(\hat{v}) = 1$, and so, $u(\mu_i(\hat{v}); v_i) = \beta - T_i^k(z)$. Now, consider a bijection $\pi : N \mapsto N$ such that $\pi i = j$, $\pi j = i$, and for all buyers $t \neq i, j$, $\pi t = t$. Hence, $(\pi\hat{v})_j^k = \beta$ and $(\pi\hat{v})_{-j}^k = z$. Therefore, by construction, $d_j^k(\pi\hat{v}) = 0$, and so, by Proposition 1, $u(\mu_j(\pi\hat{v}); (\pi\hat{v})_j) = 0$. Now, by AN, $u(\mu_i(\hat{v}); v_i) = u(\mu_{\pi i}(\pi\hat{v}); (\pi\hat{v})_{\pi i}) = u(\mu_j(\pi\hat{v}); (\pi\hat{v})_j) = 0$, implying that $\beta = T_i^k(z)$, which is a contradiction. □

Proof of necessity: Fix any mechanism $\mu \in \bar{\Gamma}_{M,N}$, any $x > 0$ and any $k \in M$. Define for any $\tau \in \{1, \ldots, n-1\}$,

$$H_x^\tau := \{z \in \mathbb{R}_{++}^{n-1} | \text{ there are } \tau \text{ coordinates of } z \text{ equal to } x \text{ and } \max_t z_t = x\ \}.$$

By Propositions 1 and 2, for any $z \in H_x^{n-1}$, and any v such that $v^k = (x, z)$, $d^k(v) \neq (0, 0, \ldots, 0)$, and so, $T^k(z) = x$.[24] Now, fix any $\bar{\tau} > 2$ and consider an induction hypothesis:

$$(\mathbf{h}) \;\; z \in \bigcup_{t=\bar{\tau}}^{n-1} H_x^t \Longrightarrow T^k(z) = x.$$

Now, consider any $z' \in H^{\bar{\tau}-1}$ such that $T^k(z') \neq x$. This leads to two possibilities: (A) $T^k(z') > x$ and (B) $T^k(z') < x$. Now, consider the profile v where $v_t^k = z_t'$ for all $t = 1, \ldots, n-1$, and $v_n^k = x$. If (A) holds true, then, by hypothesis **(h)**, Theorem 2 and Theorem 3, we get that $d_t^k(v) = 0$ for all $t \in N$, which contradicts our supposition that all objects are sold at all profiles. If (B) holds true, then, by hypothesis **(h)**, Theorem 2 and Theorem 3, we get that

$$d_n^k(v) = d_{\bar{t}}^k(v) = 1 \text{ for all } \bar{t} \neq n \text{ such that } v_{\bar{t}}^k = x,$$

which is a contradiction as each object is available in supply of one quantity only. Hence, we get that $z \in H^{\bar{\tau}-1} \implies T^k(z) = x$. Thus, we can infer that for all $z \in \bigcup_{t=1}^{n-1} H_x^t$,

$$T^k(z) = \max_t z_t = x.$$

Now, since x was arbitrarily chosen, it follows that:

$$T^k(z) = \max_t z_t, \text{ for all } z \in \mathbb{R}_{++}^{n-1},$$

[24]Given Proposition 2, henceforth, we drop the buyer identity subscript for simplicity of notation.

and so, the result follows. □

7.2. **Independence of axioms.**

7.2.1. *Theorem 5.* We use five axioms in characterizing this result: AS, AN, continuity, nonbossiness and SP. To establish independence between these axioms, we present below five mechanisms, each of which satisfy only four out of the five aforementioned properties.

AN: Say $m = 2$ and $n = 2$. Consider a mechanism where for any k, and any v,

$$T_1^k(v_2) = v_2^k + 1, T_2^k(v_1) = \max\{0, v_1^k - 1\}.$$

By Proposition 2, this mechanism does not satisfy AN. However, by Corollary 1 and Theorem 2, this mechanism is continuous and strategyproof, respectively. Further, it is easy to see that it satisfies AS and nonbossiness.

AS: Say $m = 2$ and $n = 2$. Consider a mechanism which does not sell any object to any buyer. It is easy to see that this mechanism trivially satisfies AN, nonbossiness, continuity, and SP, but does not satisfy AS.

Continuity: Say $m = 2$ and $n = 2$. Consider a mechanism where for any $i \neq j$, k, and v,

$$T_i^k(v_{-i}) = \begin{cases} 10^k & \text{if } v_j^k \in (0, 10) \\ v_j^k & \text{otherwise.} \end{cases}$$

It is easy to see that this mechanism satisfies AS, AN, SP, and nonbossiness. However, the threshold functions are not continuous, and hence, by Corollary 1, the mechanism does not satisfy continuity.

Nonbossiness: Say $m = 2$ and $n = 3$. Consider a mechanism where for any $i \neq j$, k, and v;

$$T_i^k(v_{-i}) = \begin{cases} \max_{j \neq i} v_j^k + 5 & \text{if } \max_{j \neq i} v_j^k \leq 5 \\ \max_{j \neq i} v_j^k & \text{otherwise} \end{cases}$$

It is easy to see that this mechanism satisfies AN, AS, continuity and SP. However,

$$d^1 \begin{pmatrix} 7 & 45 & 30 \\ 6 & 25 & 20 \\ 2 & 15 & 10 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = d^1 \begin{pmatrix} 7 & 45 & 30 \\ 4 & 25 & 20 \\ 2 & 15 & 10 \end{pmatrix},$$

and hence, the mechanism violates nonbossiness in decision.

SP: Say $m = 2$ and $n = 3$. Consider a mechanism that sells each object to a highest bidder i for the object at a price that is equal to the amount bid by i. It can easily be seen that this mechanism satisfies AN, AS, continuity and nonbossiness; but does not satisfy SP (as it does not belong to the class characterized by Theorem 2).

7.2.2. *Theorem 4.* We use five axioms in characterizing this result: AS, AN, continuity, no-wastage (that is, where all objects are sold at all profiles) and SP. As above, to establish independence between these axioms, we present below four example mechanisms which satisfy only four out of the five aforementioned properties. These four examples establish that neither of AS, AN, SP and no-wastage, can be obtained as an implication of the other four axioms.

AN: Say $m = 2$ and $n = 2$. Consider a mechanism where for any k, and any v,

$$T_1^k(v_2) = v_2^k + 1, T_2^k(v_1) = \max\{0, v_1^k - 1\}.$$

By Proposition 2, this mechanism does not satisfy AN. However, by Corollary 1 and Theorem 2, this mechanism is continuous and strategyproof, respectively. Further, it is easy to see that it satisfies AS and no-wastage.

AS: Say $m = 2$ and $n = 2$. Consider a mechanism which never sells any object to any buyer. It is easy to see that this mechanism trivially satisfies AN, no-wastage, continuity, and SP, but does not satisfy AS.

No-wastage: Say $m = 2$ and $n = 2$. Consider the mechanism such that for any $i \neq j$, k, v;

$$T_i^k(v_{-i}) = \max\{5, v_j\}.$$

It is easy to see that this mechanism satisfies AN, AS, continuity and SP, but does not satisfy no-wastage.

SP: Say $m = 2$ and $n = 3$. As above, consider a mechanism that sells each object to a highest bidder i for the object at a price that is equal to the amount bid by i. It can easily be seen that this mechanism satisfies AN, AS, continuity and no-wastage; but does not satisfy SP (as it does not belong to the class characterized by Theorem 2).

We are unable to present an example to rule out the possibility that any mechanism satisfying AN, AS, no-wastage and SP - would also satisfy continuity. It is our conjecture that continuity is indeed independent of these axioms.

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