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Shapley value and extended efficiency *

Ranojoy Basu[†] and Conan Mukherjee[‡]

Abstract

We formalize a new concept of '*extended efficiency*' that models important practical cooperative situations which the conventional notion of efficiency fails to accommodate. We use it to completely characterize modifications of Shapley value that satisfy monotonicity and symmetry.

Keywords: Shapley value, extended efficiency, coalitional monotonicity, marginal monotonicity, symmetry

JEL Classification: C71, D60

Introduction

As argued by the seminal work Shapley [1953], application of cooperative game theory to practical situations requires that players be able to evaluate the very "*prospect of having to play a game*". In this paper, we provide a new notion of value using an *extended* notion of efficiency along with monotonicity and symmetry axioms. This extended notion of efficiency requires the sum of individual values to exhaust, not just the grand coalitional worth, but the *sum of worths of all coalitions in a cooperative game.*

This notion of efficiency has received very little attention in the cooperative game theory literature over the years. However, it is quite intuitive and applies to several practical settings.[1] A typical example of such a setting would be a firm whose 'line workers' or 'partners'

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[1]One may think of modelling these practical settings by accounting a coalition's worth to be sum of worths of its sub-groups. However, as we argue in the Discussion section, such modelling would lead values that are socially unacceptable.

produce multiple products or services to generate profits, which in turn, are required to be redistributed as bonuses.[2] In line with Littlechild and Owen [1973], such firms can be modelled as cooperative games, where teams of workers generate coalitional worths measured as profits. The desirable bonus distribution, then, can be found from any acceptable solution to such a game. However, any such bonus distribution would be socially acceptable only if the sum of individual bonuses exhausts the sum of profits earned by *all* teams of workers; that is, the bonus distribution satisfies our notion of extended efficiency. Indeed, the same is noted about the globe-spanning behemoth 'Anderson Worldwide' with multiple departments including accounting and consultancy, by Brown and Dugan [2002], who state that:

> "*all of the profits from all the practice areas had to go into one big pot to be divided among partners*".

We combine this notion of extended efficiency with standard notions of symmetry and monotonicity to obtain a new value for a cooperative game. Our *symmetry* axiom is same as the one used by Shapley [1953], and requires that the values chosen should ignore individual identity characteristics that are exogenous to the game. We use two notions of monotonicity. The first notion of *coalition monotonicity* requires that across two different games: if every coalition containing some agent i does better in the former than the latter, then the value to i from the former game should be no less than that in the latter game. The second notion of *marginal monotonicity* is the more conventional one, first proposed by Young [1985], which requires that across any two games: if the 'marginal contribution' of any agent i in each coalition is no less in former than the latter, then the value to i should be no less in the former than the latter.[3]

Note that, apart from the technical importance in developing a value for cooperative game, the symmetry and monotonicity axioms also embody important ethical perspectives. In terms of our motivating example, the former property ensures that profits are distributed in a manner independent of personal characteristics of players endowed by nature, while the latter property embodies that natural idea that better performance should not lead to lower rewards. These properties constitute a notion of fairness that is essential for practical application of a value of a game, to a practical question of resource distribution. In absence of such fairness, any collaborative enterprise is likely to disintegrate. In fact, the aforementioned multinational company Anderson Worldwide, did break up into two separate businesses,

[2]The largest example of such a setting would a nation's economy where the gross national product is created by collaborative, and often overlapping, economic enterprises by sub-groups of citizens.

[3]As in Shapley [1953] and Young [1985], we define marginal contribution of any agent i in any coalition S, as measured by the difference $v(S) - v(S \setminus \{i\})$.

'Accenture' and 'Anderson', on account of profit sharing dispute between the accounting and consulting divisions.[4]

We obtain two values or solutions for a cooperative game. The first value is an egalitarian one which has been discussed and anticipated in the literature, albeit in different axiomatic setups. It assigns to any player i a value that is simply the sum of average worths of all coalitions containing i; and is the unique solution that satisfies extended efficiency, coalitional monotonicity, and symmetry.

The main result of our paper, however, is the second value that we obtain as the unique solution satisfying extended efficiency, marginal monotonicity, and symmetry. Note that the seminal paper Young [1985], obtained Shapley value as the unique solution satisfying conventional efficiency, marginal monotonicity, and symmetry. Thus, our second value, which is characterized by extended efficiency and the same axioms, represents the necessary modification of Shapley value to be applicable to practical settings where conventional efficiency is not applicable.

Relation to literature

Over the years, several papers have analyzed solutions for cooperative games. Most of these papers focus on obtaining elegant characterizations of the Shapley value using newer axioms. A few notable recent papers of this kind are Casajus and Huettner [2018], Casajus and Yokote [2017], Maniquet [2003] and Casajus [2011]. Casajus and Huettner [2018] introduce a notion of decomposition of value of an agent i into 'direct' and 'indirect' parts (where the latter part measures the contribution of i into direct part of other agents), and show that Shapley value is the only decomposer of the 'naive' solution (which assigns to each agent a value equal to her marginal contribution to the grand coalition). Casajus [2011] shows that Shapley value is the unique solution that conventional efficiency, null player property, and differential marginality, while Casajus and Yokote [2017] shows that the same result continues to hold (with more than two players) if one uses a weaker version of differential marginality.[5] Maniquet [2003], on the other hand, characterizes Shapley using axioms applicable to cooperative game-theroretic

[4]As noted in Brown and Dugan [2000], the overseeing arbitrator identified the institutional unfairness in extant profit redistribution process as the primary reason for this dissolution, and observed that:

> "*Andersen Consulting was right in my opinion when they claimed that the Swiss corporate entity* [Anderson Worldwide] *was not performing its coordinating obligations*".

See Nanda and Landry [1999] for further details.

[5]Note that the differential marginality axiom in Casajus and Yokote [2017] is a weaker version of the marginal monotonicity condition of Young [1985] that we use. Casajus [2018] presents a characterization of Shapley value using an even weaker condition called 'superweak' differential marginality and an additional property concerning null players.

model of queueing problems.

There are also papers that present modifications or extensions of Shapley value by either altering the underlying axioms or by imposing structures on the opportunities of cooperation.[6] One of most interesting of such modifications in recent times, is the concept of '*egalitarian Shapley values*', which are convex mixtures of Shapley value and the equal division of the grand coalition among all players. Two notable papers discussing these egalitarian Shapley values are Casajus and Huettner [2014] and Casajus and Huettner [2013] and van den Brink et al. [2013]. van den Brink et al. [2013] completely characterized this class using conventional efficiency, linearity, anonymity and weak monotonicity (a condition that is weaker than our marginal monotonicity). Casajus and Huettner [2014] characterize these solutions as the only ones that satisfy conventional efficiency, symmetry and weak monotonicity.

Our paper, too, looks for a new solution for a cooperative game that satisfies *extended efficiency* (instead of conventional efficiency) along with the standard axioms of symmetry and monotonicity. As argued above, our second solution that is developed using marginal monotonicity, presents an extension of Young's result to the idea of extended efficiency. As noted in Casajus and Huettner [2014], there are only two other such generalizations of Young's result in the transferable utility framework: Nowak and Radzik [1995] and De Clippel and Serrano [2008]. The former relaxes the symmetry assumption to present a characterization of weighted Shapley values, while the latter presents a extension of Shapley value to cooperative games with externalities (requiring the primitive to be partition function instead of characterisitic function).

With respect to our egalitarian value obtained using coalitional monotonicity, two relevant papers are: van den Brink [2007] and Moulin [1987]. The latter paper characterizes the solution that equally divides grand coalitional worth among all players in a setting where players are identified by heterogeneous opportunity costs. The former paper explores connections between the equal division of grand coalitional worth and the null player property of Shapley [1953]. It provides characterizations of this value using a modification of this null player property, and a same notion of monotonicity that is same as our axiom of coalitional monotonicity.

Note that all these papers mentioned before use the conventional axiom of efficiency. We are unaware of any other paper that uses the notion of extended efficiency axiom to present an extension of Shapley [1953].

[6] Two excellent resources surveying literature on generalizations of Shapley value in transferable utility settings are Monderer and Samet [2002] and Winter [2002].

Model

Consider a set $N = \{1, 2, \ldots, n\}$ where $n \geq 2$. For any set $S \subseteq N$, let $\rho(S)$ be the set of all possible non-empty subsets of S. Define a transferable utility cooperative game to be a pair (N, v) where N is the set of players and $v : \rho(N) \cup \emptyset \mapsto \mathbb{R}$ is a characteristic function that assigns to each possible coalition in the game a real valued worth and $v(\emptyset) := 0$. Let $\mathcal{V}(N)$ denote the class of all such characteristic functions that can be defined on the set $\rho(N)$, and define $\{(N, v)\}_{v \in \mathcal{V}(N)}$ to be the class of all possible games that can be defined on the player set N. Note that we do not impose any superadditivity restriction on the set of functions $\mathcal{V}(N)$.

Our objective is to obtain a solution (that is, a value distribution across players) for each possible game so that a society of players can make an informed choice on which games to play. That is, we seek to obtain a solution $\psi : \mathcal{V}(N) \mapsto \mathbb{R}^N$. In this paper, we require such a solution to satisfy the axioms of extended efficiency, symmetry and monotonicity.

The first axiom of extended efficiency requires that a solution should exhaust sum of worths of all coalitions. Any violation of this axiom would lead to wastage of resources, which would be unacceptable in any practical application.

Definition 1 *$\psi(\cdot)$ satisfies extended efficiency (EFF*) if and only if for all $v \in \mathcal{V}(N)$ and all $i \in N$,*

$$\sum_{i \in N} \psi_i(v) = \sum_{S \in \rho(N)} v(S).$$

Note that our notion of extended efficiency is different from the conventional notion of efficiency of a solution to a game, which requires that sum of individual values sum up to be equal to the grand coalitional worth. As argued earlier, there are several practical situations of cooperative enterprise, where this conventional efficiency is not applicable.

Our second axiom presents the basic notion of fairness that requires any solution to ignore the player identities. Shapley [1953], calls this the axiom of symmetry. To formally describe this property, we define a permutation of player identities to be a bijection $\pi : N \mapsto N$, $\pi(S)$ to be the restriction of such a permutation to any subset $S \in \rho(N)$; and $\pi v \in \mathcal{V}(N)$ to be a characteristic function derived from any other function $v \in \mathcal{V}(N)$ using this permutation $\pi(\cdot)$ of identities, where $\pi v(\pi(S)) := v(S)$ for all $S \subseteq N$.

Definition 2 *$\psi(\cdot)$ satisfies symmetry (SYM) if and only if for all bijections $\pi : N \mapsto N$ and all $i \in N, v \in \mathcal{V}(N)$,*

$$\psi_{\pi(i)}(\pi v) = \psi_i(v).$$

Our third axiom requires that the solution satisfy 'coalitional monotonicity' in the sense that the value assigned to any player i should not decrease, when the underlying characteristic function v changes to any other function v' in a manner such that worths of all coalitions containing i increases. That is, value assigned to an agent should increase if profitability of all cooperative groups containing her improves.

Definition 3 *$\psi(\cdot)$ satisfies coalitional monotonicity (C-MON) if and only if for all $v, v' \in \mathcal{V}(N)$, and all $i \in N$,*

$$[v(S \cup \{i\}) \leq v'(S \cup \{i\}),\ \forall\ S \subseteq N \setminus \{i\}] \Longrightarrow [\psi_i(v) \leq \psi_i(v')].$$

Our fourth axiom presents an alternative notion of monotonicity which requires that a solution satisfy 'marginal monotonicity' (as proposed in Young [1985]). This idea of monotonicity requires that the value assigned to any player i should not decrease, when the underlying characteristic function changes in a manner such that marginal contributions of i to all groups containing her, increase. As in Shapley [1953], we quantify such a marginal contribution of a player i to any team $S \subseteq N$, by the difference $c_v^i(S) := v(S) - v(S \setminus \{i\})$ with the convention that $c^i(\emptyset) = 0$. Further, we define for all $i \in N$ and all $v \in \mathcal{V}(N)$, the individual contribution vector $c_v^i := (c_v^i(S))_{S \subseteq N}$ as the collection of marginal contributions of each player across all coalitions in $\rho(N)$. These notations allows us to define marginal monotonicity in the following manner:

Definition 4 *$\psi(.)$ satisfies marginal monotonicity (M-MON) if and only if for all $i \in N$ and all $v, v' \in \mathcal{V}(N)$,*

$$[c_v^i \leq c_{v'}^i] \Longrightarrow [\psi_i(v) \leq \psi_i(v')].$$

Results

The following theorem states our first result. It shows that if one accepts symmetry, coalitional monotonicity and extended efficiency as the necessary properties that any solution must satisfy, then the only option available is to assign to any agent i the amount equal to *the sum of average worths of all coalitions containing i*.

Theorem 1 *A solution $\bar{\psi}(.)$ satisfies EFF*, SYM and C-MON if and only if for all $i \in N$ and all $v \in \mathcal{V}(N)$,*

$$\bar{\psi}_i(v) := \sum_{S \in \rho(N), i \in S} \frac{v(S)}{|S|}.$$

Proof: See Appendix. □

Note that Theorem 1 sums up the averages of coalitional worths to obtain the value for an individual player. Therefore, for a two player game $(\{1,2\}, v)$, Theorem 1 implies a solution:

$$\bar{\psi}_1(v) = \frac{v(12)}{2} + v(1),\ \bar{\psi}_2(v) = \frac{v(12)}{2} + v(2),$$

while for a three player game $(\{1,2,3\}, v)$, it proposes a solution:

$$\begin{aligned}
\bar{\psi}_1(v) &= \frac{v(123)}{3} + \frac{v(12)}{2} + \frac{v(13)}{2} + v(1), \\
\bar{\psi}_2(v) &= \frac{v(123)}{3} + \frac{v(12)}{2} + \frac{v(23)}{2} + v(2), \\
\bar{\psi}_3(v) &= \frac{v(123)}{3} + \frac{v(13)}{2} + \frac{v(23)}{2} + v(3).
\end{aligned}$$

Main Result

Observe that averaging of coalitional worths prior to its addition in the solution proposed by Theorem 1, lends an egalitarian character to the implied value distribution. However, it is unlikely that all members of a team put in equal amounts of efforts in generating team profits or worths. One way to account for any difference in effort or productivity of a member of a group, is to compute her marginal contribution to the team as in Shapley [1953]. The following theorem presents our main result, which states the implication of using marginal monotonicity instead of coalitional monotonicity. We find that ψ^* is the unique solution that satisfies extended efficiency, symmetry and marginal monotonicity.

Theorem 2 *A solution ψ^* satisfies EFF*, SYM and M-MON if and only if for all $i \in N$ and all $v \in \mathcal{V}(N)$,*

$$\psi_i^*(v) := \frac{1}{n!} \sum_{t=1}^{n} \beta_{n-t+1}^{n} \left\{ \sum_{\substack{S \in \rho(N), \\ |S|=t, i \in S}} c_v^i(S) \right\},$$

where $n := |N|$ and $\beta_l^n := \frac{n! + (l-1)\beta_{l-1}^n}{n-l+1}$ for all $l = 1, 2, \ldots, n$.

Proof:

Sufficiency: It can easily be seen that $\psi_i^*(\cdot)$ satisfies SYM and M-MON. To show that it satisfies EFF*, we first note that for any $v \in \mathcal{V}(N)$, any i and any $t = 1, \ldots, n$,

$$\sum_{i \in N} \sum_{\substack{S \in \rho(N), \\ |S|=t, i \in S}} c_v^i(S) \quad = t \sum_{\substack{S \in \rho(N), \\ |S|=t}} v(S) - (n-t+1) \sum_{\substack{S \in \rho(N), \\ |S|=t-1}} v(S).$$

Therefore,

$$\sum_{i \in N} \psi_i^*(v) = \frac{1}{n!} \sum_{t=1}^{n} \left\{ t\beta_{n-t+1}^n - (n-t)\beta_{n-t}^n \right\} \sum_{\substack{S \in \rho(N), \\ |S|=t}} v(S).$$

Since, $t\beta_{n-t+1}^n - (n-t)\beta_{n-t}^n = n!$ for all t; $\sum_{i \in N} \psi_i^*(v) = \sum_{S \in \rho(N)} v(S)$ and thus it satisfies EFF*. This establishes the sufficiency of the result.

Necessity: Fix a solution $\psi(\cdot)$ such that it satisfies EFF*, SYM and M-MON. Also define, for any characteristic function $v \in \mathcal{V}(N)$,

$$\eta(v) := |\rho(N)| - |\{S \in \rho(N) : v(S) = 0\}| + 1.$$

Now, consider a characteristic function $v \in \mathcal{V}(N)$ with $\eta(v) = 1$. For all $i \neq j$ and all permutations π such that $\pi(i) = j, \pi(j) = i$, SYM implies that $\psi_i(v) = \psi_j(v)$. Thus, by EFF*, we get that for any characteristic function v with $\eta(v) = 1$, $\psi_i(v) = 0 = \psi_i^*(v)$ for all $i \in N$. Now, fix a $k \in \{1, \ldots, |\rho(N)|\}$ and suppose that for all $v \in \mathcal{V}(N)$ such that $\eta(v) \leq k$, $\psi_i(v) = \psi_i^*(v)$ for all i. In the following paragraphs, we show how this induction hypothesis implies that for all $v \in \mathcal{V}(N)$ such that $\eta(v) = k+1$, $\psi_i(v) = \psi_i^*(v)$ for all $i \in N$.

Fix a $v \in \mathcal{V}(N)$ with $\eta(v) = k+1$ and any $i \in N$. If there exist a $T_i \in \rho(N)$ such that $i \notin T_i$ and $v(T_i) \neq 0$, then construct a characteristic function $v_{T_i} \in \mathcal{V}(N)$ such that for all

$$v_{T_i}(S) := \begin{cases} v(T_i \cup \{i\}) - v(T_i) & \text{if } S = T_i \cup \{i\} \\ 0 & \text{if } S = T_i \\ v(S) & \text{otherwise} \end{cases}$$

By M-MON and induction hypothesis, $\psi_i(v) = \psi_i(v_{T_i}) = \psi_i^*(v_{T_i}) = \psi_i^*(v)$.

Now, consider the other possibility where for all $T \in \rho(N)$ with $i \notin T$, $v(T) = 0$. This possibility leads to two further cases: (i) $v(\{i\}) \neq 0$ and (ii) $v(\{i\}) = 0$. In case (i), construct the characteristic function $\tilde{v} \in \mathcal{V}(N)$ such that for all $S \in \rho(N)$; if $i \notin S$, $\tilde{v}(S) := v(S)$ and if $i \in S$, $\tilde{v}(S) := v(S) - v(\{i\})$. By M-MON and induction hypothesis, for all $j \neq i$, $\psi_j(v) = \psi_j(\tilde{v}) = \psi_j^*(\tilde{v}) = \psi_j^*(v)$. Since $\psi^*(.)$ satisfies EFF* (as shown in the proof of sufficiency), it follows that $\psi_i(v) = \psi_i^*(v)$.

In case (ii), define a set $\{S_1, S_2, \ldots, S_t\} := \{S \in \rho(N) | v(S) \neq 0, S \neq N\}$. Since $\eta(v) = k+1$, such a set is well defined. Further, define $E := \cap_{r=1}^{t} S_r$ and note that $i \in E$ by construction.[7] If $|E| > 1$, then fix any $j \neq k \in E$, and consider the bijection $\pi^{jk} : N \mapsto N$ such that $\pi^{jk}(j) = k$, $\pi^{jk}(k) = j$, and $\pi^{jk}(l) = l$ for all $l \in N \setminus \{j, k\}$. By construction, for any $T \subseteq N$, if $E \subseteq T$, then $\pi^{jk}(T) = T$ which implies that $\pi^{jk}v(T) = \pi^{jk}v(\pi^{jk}(T)) = v(T)$;[8] or else (that is, if E is not a subset of T), $\pi^{jk}v(T) = v(T) = 0$. Therefore, by SYM, it follows that whenever $|E| > 1$, $\psi_j(v) = \psi_k(v)$, $\forall \{j, k\} \subseteq E$. Thus, EFF* implies that for all $j \in E$, $\psi_j(v) = \frac{1}{|E|}\left\{\sum_{r=1,\ldots t} v(S_r) + v(N) - \sum_{l \notin E} \psi_l(v)\right\}$, which is equivalent to the statement (**a**) $\psi_j(v) = \frac{1}{|E|}\left\{\sum_{\substack{i \in S,\\ S \subseteq N}} v(S) - \sum_{l \notin E} \psi_l(v)\right\}, \forall\, j \in E$. Also note that if $|E| = 1$, then (**a**) follows directly from EFF*.

To compute $\psi_l(v)$ for any $l \notin E$, note that for any such agent l, by construction, there exists an $S_{r^l} \in \{S_1, \ldots, S_t\}$ such that $l \notin S_{r^l}$. Define the characteristic function $\hat{v}_{r^l} \in \mathcal{V}(N)$ such that $\forall\, T \in \rho(N)$; if $S_{r^l} \subseteq T$, $\hat{v}_{r^l}(T) := v(T) - v(S_{r^l})$, or else $\hat{v}_{r^l}(T) := v(T)$. Now, by M-MON and induction hypothesis, for all $j \in N \setminus S_{r^l}$, $\psi_j(v) = \psi_j(\hat{v}_{r^l}) = \psi^*_j(\hat{v}_{r^l}) = \psi^*_j(v)$. Thus, arguing in this manner, we can show that for all $l \notin E$, $\psi_l(v) = \psi_l(\hat{v}_{r^l}) = \psi^*_l(\hat{v}_{r^l}) = \psi^*_l(v)$. Therefore, for all $j \in E$, $\psi_j(v) = \frac{1}{|E|}\left\{\sum_{\substack{j \in S\\ S \subseteq N}} v(S) - \sum_{\in E} \psi^*_l(v)\right\}$. Further, by construction of E, $\sum_{\substack{j \in S\\ S \subseteq N}} v(S) = \sum_{S \in \rho(N)} v(S)$ for all $j \in E$. So, by the proof of sufficiency, EFF* implies that $\psi_j(E) = \frac{1}{|E|} \sum_{l \in E} \psi^*_l(v)$. It is easy to check that $\psi^*_l(v) = \psi^*_{l'}$ for all $l, l' \in E$, and so, we get that $\psi_j(v) = \psi^*_j(v)$ for all $j \in E$. □

It can be seen that Theorem 2 prescribes for two player game $(\{1, 2\}, v)$, a solution where,

$$\begin{aligned}
\psi^*_1(v) &= \frac{v(12) - v(2)}{2} + \frac{3v(1)}{2} \\
\psi^*_2(v) &= \frac{v(12) - v(1)}{2} + \frac{3v(2)}{2},
\end{aligned}$$

[7] Recall that we are considering the possibility where for all $T \in \rho(N)$ with $i \notin T$, $v(T) = 0$.
[8] See definition of the characteristic $\pi v(.)$ in Definition 2.

while for a three player game $(\{1,2,3\},v)$, it proposes a solution:

$$\begin{aligned}
\psi_1^*(v) &= \frac{v(123)-v(23)}{3} + \frac{2\{[v(13)-v(3)]+[v(12)-v(2)]\}}{3} + \frac{7}{3}v(1) \\
\psi_2^*(v) &= \frac{v(123)-v(23)}{3} + \frac{2\{[v(13)-v(3)]+[v(12)-v(2)]\}}{3} + \frac{7}{3}v(2) \\
\psi_3^*(v) &= \frac{v(123)-v(12)}{3} + \frac{2\{[v(13)-v(3)]+[v(12)-v(2)]\}}{3} + \frac{7}{3}v(3).
\end{aligned}$$

Note how, unlike Theorem 1, Theorem 2 requires that the weights given to marginal contribution of any player to groups containing her, decrease as the group sizes increase. So the least weight is given to the marginal contribution to the grand coalition, while the maximum weight is given to the singleton coalition (that is, what the player can do alone).

Note that a difficult feature of functional form of the value $\psi^*(.)$ presented in Theorem 2 is that the coefficients $\beta_1^n, \beta_2^n, \ldots, \beta_n^n$ are defined in a recursive manner. The following corollary presents a simpler functional formulation of the β_t values.

Corollary 1 *For any* $t = 1, \ldots, n$,

$$\beta_t^n = (n-t)!(t-1)! \sum_{k=0}^{t-1} \binom{n}{k}.$$

Proof: We prove this result by induction. Note that $\beta_1^n = \frac{n!+(1-1)\beta_0^n}{n-1+1} = (n-1)!\ 0!\ \binom{n}{0}$. Now suppose that for all $m \in \mathbb{N}$, $\beta_m^n = (n-m)!(m-1)! \sum_{k=0}^{m-1} \binom{n}{k}$. Then, by Theorem 2,

$$\begin{aligned}
\beta_{m+1}^n &= \tfrac{n!+(m+1-1)\beta_m^n}{n-(m+1)+1} \\
&= \tfrac{n!+m(m-1)!(n-m)!\sum_{k=0}^{m-1}\binom{n}{k}}{n-m} \\
&= m!(n-m-1)!\left\{\tfrac{n(n-1)\ldots(n-m+1)}{m!} + \textstyle\sum_{k=0}^{m-1}\binom{n}{k}\right\} \\
&= \{(m+1)-1\}!\{n-(m+1)\}!\textstyle\sum_{k=0}^{\{(m+1)-1\}}\binom{n}{k}
\end{aligned}$$

and so, the result follows. □

Therefore, in light of Corollary 1, for any $i \in N$ and $v \in \mathcal{V}(N)$, ψ_i^* can be rewritten as follows:

$$\psi_i^*(v) := \sum_{S \in \rho(N)} \gamma_s^* c_v^i(S)$$

where for all $t = 1\ldots, n$, $\gamma_t^* := \frac{\beta_{n-t+1}^n}{n!} = \frac{1}{\binom{n-1}{t-1}} \sum_{k=0}^{n-t} \frac{1}{n-k}\binom{n-1}{k}$ and $s := |S|, \forall\ S \in \rho(N)$. Thus, Corollary 1 allows us to represent $\psi_i^*(\cdot)$ as a linear combination of marginal contribution of player i, with the weights being given by γ_s^*.

The following example provides a contrast between the two values presented by Theorems 1 and 2, by applying them to the contentious, but relevant, problem of bonus distribution which led to dissolution of Anderson Worldwide.

Example 1 Consider a problem with 3 line workers who have collaborated in pairs, as well as a three member group to service three different clients over the year, and have generated aggregate profit of $130. Their performance numbers are as shown in the following table 1.

S	{1}	{2}	{3}	{1, 2}	{1, 3}	{2, 3}	{1, 2, 3}
v(S)	0	0	0	10	20	30	70

Table 1: *Team performance data.*

Given this data, as in Shapley [1953], we can compute individual contributions of each worker to each team by calculating for all $i = 1, 2, 3$, $c_v^i(S) = v(S) - v(S \setminus \{i\})$. This data is presented in the following Table 2.

S	$v(S)$	$c_v^1(S)$	$c_v^2(S)$	$c_v^3(S)$
{1}	0	0	0	0
{2}	0	0	0	0
{3}	0	0	0	0
{1, 2}	10	10	10	0
{1, 3}	20	20	0	20
{2, 3}	30	0	30	30
{1, 2, 3}	70	40	50	60

Table 2: *Individual contributions to each team.*

Thus, we can obtain a bonus distribution as per the two rules that we have presented in the following Table 3:

Discussion

A case can be made for an alternative manner of cooperative game theory modelling of a practical social setting using the following modified characteristic function $w(\cdot)$, which

	1	2	3
$\bar{\psi}$ (*Theorem 1*)	$\frac{115}{3}$	$\frac{130}{3}$	$\frac{145}{3}$
ψ^*(*Theorem 2*)	$\frac{100}{3}$	$\frac{130}{3}$	$\frac{160}{3}$

Table 3: *Bonus distributions.*

assigns worth of any group of players $S \subseteq N$ to be $w(S) := \sum_{T \subseteq S} v(s)$ (that is, worth of S is sum of worths of all subsets T of S). For such a model, the conventional notion of efficiency would imply our extended efficiency. In terms of the example discussed above, the modified characteristic function becomes as described in table 4.

S	$w(S)$
{1}	0
{2}	0
{3}	0
{1, 2}	10
{1, 3}	20
{2, 3}	30
{1, 2, 3}	130

Table 4: *Modified characteristic function.*

Note that the Shapley value for this modified game for player i given by

$$\varphi_i^S(w) := \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n-|S|-1)!}{n!}(w(S \cup \{i\}) - w(S)), \tag{1}$$

where $N = \{1, 2, 3\}$, and $\varphi_i^S(w)$ represents the bonus received by player i. Simple calculation using the modified characteristic function in table 4 yields the Shapley distribution of bonuses: $\varphi_1^S(w) = \frac{115}{3}$, $\varphi_2^S(w) = \frac{130}{3}$, $\varphi_3^S(w) = \frac{145}{3}$. Interestingly, this is the same distribution as the one prescribed by Theorem 1.

It may, therefore, appear that the Shapley [1953] value for this modified game $(w(\cdot), N)$ would lead to same profit distribution (see table 3) as the distribution rule $\psi^*(\cdot)$ described in Theorem 2, since the latter relies on the same notion of marginal contribution as Shapley [1953]. However, this is *not* the case as the profit distribution implied by Theorem 2 is $\left(\frac{100}{3}, \frac{130}{3}, \frac{160}{3}\right)$, which is different from the Shapley value for the modified game $\left(\frac{115}{3}, \frac{130}{3}, \frac{145}{3}\right)$.

Most importantly, however, this alternate manner of constructing a characteristic function may be socially *unacceptable* in a practical setting, as the value generated by a group of players gets attributed to a larger set. That is, for a simple two player game $(\{1,2\}, v)$: the marginal contribution of 1 is now $v(2) + v(12)$, which is unlikely to be acceptable to the player 2, who finds that working harder on her own enhances the marginal contribution of her competitor within the organization. Similarly, player 1 would find it difficult to accept such accounting procedure where her marginal contribution vector depends on the individual performance of her competitor. Hence, from an application perspective to real life problems, potentially of great importance in a country's economy, our approach of constructing a characteristic function is more useful.[9]

Conclusion

In this paper, we formalize a novel notion of extended efficiency to conceptualize the no-wastage condition in settings where the traditional notion of efficiency is not applicable. Such settings are those where the members of a society work in sub-groups to generate resources for the society; like the gross national product of a nation being generated by various cooperative enterprises among sub-groups of her citizens. Unlike the conventional efficiency axiom of cooperative game theory literature, this axiom requires that a solution to a game assign individual values that sum up to equal the sum of worths of all possible coalitions.

We use this novel axiom, along with the standard monotonicity and symmetry axioms to characterize a new solution for cooperative games which, in the spirit of Young [1985], presents an extension of Shapley value to these practical settings.

Appendix

Independence of Axioms

Theorem 1

For simplicity of exposition, consider a 2-player game $(N = \{1,2\}, v)$. Clearly there are three possible coalitions: $\{1\}$, $\{2\}$ and $\{1,2\}$. Consider the following solutions:

[9] As noted in Mossin [1968], these considerations are applicable to situations like firm mergers, which may be as critical as the famous merger of Merril Lynch with Bank of America in wake of the sub-prime crisis of 2008 that led to saving of thousands of jobs.

- $\psi_1(v) = v(\{1\}) + 0.8v(\{1,2\})$, $\psi_2(v) = v(\{2\}) + 0.2v(\{1,2\})$. It is easy to see that this solution satisfies EFF* and C-MON. However, if the agent labels were interchanged, the individual values would not get interchanged for all possible $v(.)$ - implying that this rule does not satisfy SYM.

- $\psi_1(v) = v(\{1\}) + \frac{v(\{1,2\})}{4}$, $\psi_2(v) = v(\{2\}) + \frac{v(\{1,2\})}{4}$. It is easy to see that this solution satisfies C-MON and SYM. However, for any $v(.)$, $\psi_1(v) + \psi_2(v) = v(\{1\}) + v(\{2\}) + \frac{v(\{1,2\})}{2}$, and so, this rule does not satisfy EFF*.

- $\psi_1(v) = v(\{1\}) + \frac{v(\{1\})\,v(\{1,2\})}{v(\{1\})+v(\{2\})}$, $\psi_2(v) = v(\{2\}) + \frac{v(\{2\})\,v(\{1,2\})}{v(\{1\})+v(\{2\})}$. It is easy to see that this rule satisfies EFF* and SYM. However, consider two characteristic functions, $w(.)$ and $w'(.)$ such that $w(\{1,2\}) = w'(\{1,2\})$, $w(\{1\}) = w'(\{1\})$ and $w(\{2\}) > w'(\{2\})$. It is easy to see that $\psi_1(w) < \psi_1(w')$ even though 1's coalitional worths in the groups $\{1\}$ and $\{1,2\}$ remain unchanged across characteristic functions w and w'. Note that, by C-MON,

$$[w(\{1\}) = w'(\{1\}), w(\{1,2,\}) = w'(\{1,2,\})] \implies \psi_1(w) = \psi_1(w'),$$

  and so, this solution violates C-MON.

Theorem 2

As before, for simplicity of exposition, we consider a 2 player game $(N = \{1,2\}, w)$ with three possible coalitions: $\{1\}$, $\{2\}$ and $\{1,2\}$. Consider the following solutions:

- $\psi'_1(v) = 0.75v(\{1\}) + 0.25(v(\{1,2\}) - v(\{2\}))$, $\quad \psi'_2(v) = 0.75v(\{2\}) + 0.25(v(\{1,2\}) - v(\{1\}))$. It is easy to see that this solution satisfies M-MON and SYM. However, $\psi'_1(v) + \psi'_2(v) = 0.5[v(\{1,2\}) + v(\{1\}) + v(\{2\})]$, and so, it does not satisfy EFF*.

- Fix a small enough $\epsilon > 0$, and consider $\psi'_1(v) = 1.5v(\{1\}) + 0.5(v(\{1,2\}) - v(\{2\})) + \epsilon$, $\quad \psi'_2(v) = 1.5v(\{2\}) + 0.5(v(\{1,2\}) - v(\{1\})) - \epsilon$. It is easy to see that this rule satisfies EFF* and M-MON but does not satisfy SYM (as an interchange of agent labels would not lead to interchange in individual values).

- $\psi'_1(v) = v(\{1\}) + \frac{v(\{2\})\,v(\{1,2\})}{v(\{1\})+v(\{2\})}$, $\psi'_2(v) = v(\{2\}) + \frac{v(\{1\})\,v(\{1,2\})}{v(\{1\})+v(\{2\})}$. It is easy to see that this solution satisfies EFF* and SYM. However, consider two characteristic functions v and v' such that $v(\{2\}) > v'(\{2\})$ and $v(S) = v'(S)$ when $S \in \{\{1\},\{1,2\}\}$. This means that $c^1_v(\{1\}) = c^1_{v'}(\{1\})$, and $c^1_v(\{1,2\}) < c^1_{v'}(\{1,2\})$. Therefore, M-MON requires that $\psi'_1(v) \leq \psi'_1(v')$. However, by construction, $\psi'_1(v) > \psi'_1(v')$, and so, it follows that this solution violates M-MON.

Proof of Theorem 1

Define for all $i \in N$ and all $v \in \mathcal{V}(N)$, $\bar{\psi}_i(v) := \sum\limits_{S \in \rho(N), i \in S} \frac{v(S)}{|S|}$.[10]

It can easily be checked that $\bar{\psi}_i(v)$ satisfies EFF*, SYM and C-MON, and so, the proof of sufficiency follows. To prove necessity, fix any solution $\psi(.)$ satisfying EFF*, SYM and C-MON, and any $i \in N$. Now consider the partition of $\mathcal{V}(N)$ into the set $\mathcal{P} := \{V^0, V^2, \ldots, V^{|\rho(N)|}\}$ such that for all $k = 0, \ldots, |\rho(N)|$, V^k is the set of characteristic functions such that there are exactly k teams in $\rho(N)$ who have posted zero profit/worth. It can easily be seen that: (i) by construction $\mathcal{V}(N) = \mathbb{R}_+^{|\rho(N)|}$, (ii) for all $k \neq l \in \{1, \ldots, n\}$, $V^k \cap V^l = \emptyset$, and $\cup_{k=0}^n V^k = \mathbb{R}_+^{|\rho(N)|}$. Hence, $\mathcal{P}$ is well defined.

Now fix any characteristic function $v^{|\rho(N)|} \in V^{|\rho(N)|}$, any $i \neq j$, and any permutation $\pi^{ij} : N \mapsto N$ such that $\pi^{ij}(i) = j, \pi^{ij}(j) = i$. Note that by SYM, $\psi_i(v^{|\rho(N)|}) = \psi_j(v^{|\rho(N)|})$. Hence, EFF* implies that $n\psi_i(v^{|\rho(N)|}) = 0 \implies \psi_i(v^{|\rho(N)|}) = 0, \forall\, i \in N$. Now suppose that for some $l \in \{1, \ldots, |\rho(N)|\}$, (**a**) $v^l \in V^l \implies \psi_i(v^l) = \bar{\psi}_i(v^l), \forall\, i \in N$. Now consider any $v^{l-1} \in V^{l-1}$, and define $\tilde{N}(v^{l-1}) := \{i \in N | \forall\, S \in \rho(N), i \notin S \implies v^{l-1}(S) = 0\}$. Thus $\tilde{N}(v^{l-1}) \subseteq N$ is the set of agents i such that any team $S \in \rho(N)$ not containing i, has zero worth in v^{l-1}. Therefore, if $\tilde{N}(v^{l-1}) = N$, then for any $S \in \rho(N)$, $S \neq N \implies v^{l-1}(S) = 0$, and so, $v^{l-1}(N) > 0$ (as $l - 1 < |\rho(N)|$ by construction). Now, as before, any $i \neq j \in N$, and any permutation π^{ij} with $\pi^{ij}(i) = j$ and $\pi^{ij}(j) = i$, $\pi^{ij}v^{l-1} = v^{l-1}$, and so, by SYM, $\psi_i(v^{l-1}) = \psi_j(v^{l-1})$. Hence, EFF* implies that $\psi_i(v^{l-1}) = \bar{\psi}_i(v^{l-1})$. This establishes the result for the case where $\tilde{N}(v^{l-1}) = N$.

Now, if $\tilde{N}(v^{l-1}) \subset N$, then for any $i \notin \tilde{N}(v^{l-1})$, choose a $T^i(v^{l-1}) \in \rho(N)$ such that $i \notin T^i(v^{l-1})$ and $v^{l-1}(T^i(v^{l-1})) > 0$. Note that by construction of $\tilde{N}(v^{l-1})$, the set $T^i(v^{l-1})$ is well defined. Construct a characteristic function $\tilde{v}_i^l \in V^l$ where for all $S \in \rho(N)$, $S \neq T^i(v^{l-1}) \implies \tilde{v}^l(S) = v^{l-1}(S)$ and $\tilde{v}^l(T^i(v^{l-1})) = 0$. By supposition (**a**) and C-MON, $\psi_i(v^{l-1}) = \psi_i(\tilde{v}^l) = \bar{\psi}_i(\tilde{v}^l)$ for all $i \notin \tilde{N}(v^{l-1})$. This establishes the result for the case where $\tilde{N}(v^{l-1}) = \emptyset$. Further, if $|\tilde{N}(v^{l-1})| = 1$, that is, supposing $\tilde{N}(v^{l-1}) = \{l^*\}$, by EFF*, $\psi_{l^*}(v^{l-1}) = \sum_{S \in \rho(N)} v(S) - \sum_{i \notin \tilde{N}(v^{l-1})} \bar{\psi}(v^{l-1})$ which equals $\bar{\psi}_{l^*}(\tilde{N}(v^{l-1})$, because as argued in proof of sufficiency above, $\bar{\psi}(.)$ satisfies EFF*.

Now, to establish the result for the only remaining possibility where $0 < |\tilde{N}(v^{l-1})| < n$, note that by construction, for any $S \in \rho(N)$, $v^{l-1}(S) > 0 \implies \tilde{N}(v^{l-1}) \subseteq S$. Therefore, for any $i \neq j \in \tilde{N}(v^{l-1})$, and any permutation π^{ij} such that $\pi^{ij}(i) = j, \pi^{ij}(j) = i$, $\pi^{ij}v^{l-1} = v^{l-1}$, and so, by SYM, $\psi_i(v^{l-1}) = \psi_j(v^{l-1})$. Therefore, EFF* implies that for all $i \in \tilde{N}(v^{l-1})$, $|\tilde{N}(v^{l-1})|\psi_i(v^{l-1}) = \sum_{S \in \rho(N)} v^{l-1}(S) - \sum_{j \notin \tilde{N}(v^{l-1})} \psi_j(v^{l-1}) = \sum_{S \in \rho(N)} v^{l-1}(S) -$

[10] The proof technique resembles a similar result is proved in Mukherjee et al. [2020].

$\sum_{j \notin \tilde{N}(v^{l-1})} \bar{\psi}_j(v^{l-1})$. Therefore,[11]

$$\psi_i(v^{l-1}) = \frac{1}{|\tilde{N}(v^{l-1})|} \left(\sum_{\tilde{N}(v^{l-1}) \subseteq S} v^{l-1}(S) - \sum_{j \notin \tilde{N}(v^{l-1})} \sum_{\substack{j \in S, \\ \tilde{N}(v^{l-1}) \subseteq S}} \frac{v^{l-1}(S)}{|S|} \right),$$

and so, for all $i \in \tilde{N}(v^{l-1})$, $\psi_i(v^{l-1}) = \frac{1}{|\tilde{N}(v^{l-1})|} \sum_{\tilde{N}(v^{l-1}) \subseteq S} \left\{ v^{l-1}(S) - \sum_{j \in S \setminus \tilde{N}(v^{l-1})} \frac{v^{l-1}(S)}{|S|} \right\} = \frac{1}{|\tilde{N}(v^{l-1})|} \sum_{\tilde{N}(v^{l-1}) \subseteq S} \frac{|\tilde{N}(v^{l-1})| v^{l-1}(S)}{|S|}$, which by construction of $\tilde{N}(v^{l-1})$, is equal to $\sum_{\substack{i \in S, \\ S \in \rho(N)}} \frac{v^{l-1}(S)}{|S|} = \bar{\psi}_i(v^{l-1})$. □

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[11] Recall that, by construction, for any $S \in \rho(N)$ that does not contain $\tilde{N}(v^{l-1})$, $v^{l-1}(S) = 0$.

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