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ABSTRACT:*This paper studies the performance of Heston Model and Black-Scholes Model in pricing index options. I have compared the two models based on 1074 call option prices of S&P 500 on1st November, 2016. I have calibrated the parameters of the Heston Model by non-linear least square optimization using call option prices from a period of 20 days (3rd October, 2016 to 31st October, 2016). The in-sample data had a total of 25,392 call options and thus 20 strike prices for each time-to-maturity. We observe that both Heston Model and Black Scholes Model under-price in-the-money options and over-price out-of-the money options, but the degree of error is different.Black Scholes Model slightly outperforms Heston Model for short term ITM, DITM and ATM options where Heston Model is unable to capture the high implied volatility.But Heston Model starts to give better estimates for ITM, DITM and ATM options as the time-to-maturity increases. For OTM and DOTM options, Heston Model significantly outperforms Black Scholes Model. In most of the cases, the implied volatility calculated from Heston model prices is found to be less than that calculated from market prices for different combinations of moneyness and time-to-maturity.*

1. INTRODUCTION

The Black Scholes model (1973) is frequently used to price European options. But this model is based on various assumptions which are not representative of real world financial markets. It assumes that the volatility remains constanttill maturity of the options. Due to the volatility skewof equity options, Black-Scholes (1973) formula tends to misprice OTM and ITM options if the VIX value is used. During the last decades, several alternatives have been proposed to model volatility for pricing options. One such approach is introducing uncertainty in the behaviour of volatility, i.e., making volatility a stochastic quantity. By estimating the parameters of a stochastic process, it can be used to estimate prices close to the market values.One of the most widely used stochastic volatility modelsto price options was proposed by Heston (1993).

The paper is structured as follows: In Section 2, I give a brief background and literature review of option pricing models. In Section 3, I present the valuation framework of Heston and Black Scholes model and how characteristic equations can be used to price the options. In Section 4, I elaborate on the methodology of calibrating the Heston parameters using Local Optimization method.

Finally,in Section 5, I estimate the option prices using the calibrated parameters and analyse the results.

2. LITERATURE REVIEW

In 1900, Louis Bachelier introduced the concept of Brownian motion in financial markets. In 1973, Fischer Black and Myron Scholes proposed a famous model, based on Geometric Brownian Motion, for pricing European options. In a Geometric Brownian motion, the logarithm of the randomly varying quantity, for example, stock price follows a Brownian motion. It assumes that the volatility of stock returns remains constantand the distribution of logarithmic returns is normal. But the 1987 crash revealed some of the shortcomings of the Black-Scholes Model. In fact, the returns exhibit skewness and kurtosis which is not considered in the model. When the volatility surface is plotted using the implied volatility from Black Scholes equation with respect to time-to-maturity and strike price of options, it is flat. But in reality, the implied volatility surface is skewed, i.e., the volatility is different for various combinations of strike prices and time-to-maturity. This disparity led to many attempts to create models which will estimate the option prices better. A real breakthrough came when Steven Heston (1993) incorporated stochastic volatility into the option pricing model. Unlike Black Scholes model where only the stock price followed a stochastic process, Heston model had two stochastic processes, one for the stock price and the other for the volatility. Bates (1996) extended the model by incorporating jumps in the stock price dynamics, which is useful for pricing out-of-the-money options. But as the complexity of the model increases, more parameters need to be estimated which may not be useful in real life. In this paper, I will use Heston's model to estimate the option prices and compare it with Black Scholes prices.

3. VALUATION FRAMEWORK

The Black Scholes price can be calculated without using characteristic functions**.** But Heston model has two stochastic processes and these processes have characteristic functions which are easier to code. Therefore, I will use the characteristics functions instead of density functions to calculate Heston prices.

3.1. BLACK SCHOLES MODEL

The risk-neutral dynamics of an assetcan by described by$dS_t = rS_t dt + \sigma_t S_t dW_t$

where S_t is the price of the asset, i.e., the index level at time t, r is the risk-free rate, σ_t is the return volatility, and W_t is a Brownian process. From this, we can derive

$$S_t = S_0 e^{(r-0.5\sigma^2)t+\sigma\sqrt{t}Z}$$

where Z is the standard normal distribution. The Black Scholes Price can be calculated as

$$C = S_0 N(d_1) - Ke^{-rT}N(d_2)$$

Where $N(d_1) = \frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $N(d_2) = N(d_1) - \sigma\sqrt{T}$

3.2. HESTON MODEL

In 1993, Heston proposed a stochastic volatility model where both volatility and underlying asset follow stochastic processes. These can be described as follows:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^1$$

$$dV_t = \alpha(\bar{V} - V_t)dt + \eta\sqrt{V_t}dW_t^2$$
$$dW^1 dW^2 = \rho dt$$

Where, S_t is the price of the index level in this case at time t, r is the risk-free rate r, V_t is the variance at time t, $\bar{V}$ is the long-term variance V, α is the variance mean-reversion speed, η is the volatility of the variance process, W_t^1 and W_t^2 are two correlated Brownian motion and ρ is the correlation coefficient.

The price of a European call option can be obtained by using the following equation:

$$C_0 = S_0\Pi_1 - e^{-rT}K\Pi_2$$

where Π_1 is the delta of the option and Π_2 is the risk-neutral probability of exercise (i.e. when $S_T>K$)

For j=1,2 the Heston characteristic function is given as

$$f_j(x, v, t;\ \phi) = e^{C(t-t;\phi)+D(T-t;\phi)v+i\phi x}$$

where

$$C(\tau;\phi) = r\phi i\tau + \frac{\mathrm{a}}{\sigma^2}\left\{(b_j - \rho\sigma\phi i + d)\tau - 2\ln\left[\frac{1-ge^{d\tau}}{1-g}\right]\right\}$$

$$D(\tau;\phi) = \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left[\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right]$$

$$g = \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}$$

$$d = \sqrt{\left(\rho\sigma\phi i - b_j\right)^2 - \sigma^2\left(2u_f\phi i - \phi^2\right)}$$

$$u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, a = \kappa\theta, \mathrm{b}_1 = \kappa - \rho\sigma, \mathrm{b}_2 = \kappa$$

The characteristic functions can be inverted to get the required probabilities

$$\Pi_{\mathrm{j}}(x, v, T; \ln[K]) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\ln[K]}f_j(x, v, T;\ \phi)}{i\phi}\right]d\phi$$

4.METHODOLOGY

From the in-sample data, I have calibrated the Heston parameters and using these parameters, I have estimated the options prices for out-of-sample period.

4.1 DATA

I have collected the call option data for S&P 500 index from ivolatility.com. I have considered the in-sample period as 3rd October to 31st October, 2016 and out-of-sample period as 1st November, 2016. First, I have removed the arbitrage violations from the data to avoid anynegative impliedvolatility. A call option price must satisfy $C \geq \max(0, S - D - Ke^{-rt})$ to remove arbitrage possibilities. Second, I have excluded the options with less than 7 days to expiration or more than 180 days to expiration because they are very sensitive to liquidity-related biases. Third, I have excluded very deep out-of-the-money and very deep in-the-money options, because they are not liquid options and their market prices may be quite different from their true values. An option is very deep in-the-money if its moneyness is greater than 9% and very deep out-of-the-money if its moneyness is less than-9%.

The option moneyness is defined as the percentage difference between the current underlyingprice and the strike price: Moneyness(%) = $S / K - 1$

The in-sample data has 25,392 call options for 20 days (i.e., 20 strike prices for each maturity). The out-of-sample initially had 1,236 call options. But since some Heston prices and implied volatilities were negative, I removed these out-of-sample data points to create arbitrage free option prices. Finally, I was left with 1074 call options in the out-of-sample.

4.2 CALIBRATION TO MARKET PRICES

I intend to determine the set of parameters which minimizes the distance between model prices and market prices. The Heston model has five unknown parameters i.e., initial variance, long-term variance, correlation between the two stochastic processes, volatility of variance and mean reversion speed.

In order to find the optimal parameter-set, we need to minimize the mean sum of squared differences

$$G(\Omega) = \sum_{i=1}^{N} \frac{1}{N} [C_i^{\Omega}(K_i, T_i) - C_i^{Mkt}(K_i, T_i)]\text{^}2$$

Where $C_i^{\Omega}\ (K_i, T_i)$ are the option prices using the Heston parameter set Ω and $C_i^{Mkt}(K_i, T_i)$ are the market observed option prices.

Initial variance: Bounds of 0 and 1 have been used. Volatility above 100% is quite unrealistic.

Long-term variance: Same bounds as above have been used.

Correlation:Correlation between the stochastic processes (volatility and underlying price) takes values from -1 to 1. Although, the correlation is usually negative, positive correlations are also possible.

Volatility of variance:It exhibits positives values. Since the volatility of assets may increase in the short term,a broad range of 0 to 5 will be used.

Mean-reversion speed:This will will be dynamically set using a non-negativity constraint (Feller, 1951).The constraint $2\alpha\bar{V} - \eta^2 > 0$ guarantees that the variance in a CIR process is always strictly positive.

Local Optimization: For local optimization, the MATLAB function lnsqnonlin will be used.

Heston Calibrated Parameters under non-linear least square optimization

V_0	$\bar{V}$	η	ρ	α
0.0112	0.0156	0.5010	-0.9723	168.1191

Initial volatility = $\sqrt{V_0}$ = 10.58%

Long term volatility = $\sqrt{V_t}$ = 12.49%

Volatility of the variance process = 50.10%

Correlation Coefficient = -0.9723

Mean Reversion Speed = 168.1191

5. RESULTS AND ANALYSIS

The outputs have been divided in terms of moneyness and time-to-maturity.

ATM – moneyness lies between -2% and 2%

ITM – moneyness lies between 2% and 6%

OTM – moneyness lies between -2% and -6%

DITM – moneyness is greater than 6%

DOTM – moneyness is less than -6%

$$MRPE = \frac{1}{N}\sum_{I=1}^{N}\frac{|model\ price - market\ price|}{market\ price}$$

Implied volatility has been calculated by equating the market price or the Heston price to the Black Scholes price. I have used the MATLAB function blsimpv to calculate the implied volatilities.

In-the-Money Call Options

MRPE

Maturity	<=45 days	45-90 days	>90 days
BSM	0.0224	0.0374	0.0686
Heston (lnsqnonlin)	0.0714	0.0703	0.0560

Implied Volatility

Maturity	<=45 days	45-90 days	>90 days
Market	17.76%	15.86%	15.10%
Heston (lnsqnonlin)	12.75%	13.16%	13.16%

In this case, the Black Scholes model outperforms the Heston Model for short term ITM options, because Heston model is not able to capture the short term high volatility. As time-to-maturity increases, Heston starts to give better estimates and finally outperforms Black Scholes model for maturity greater than 90 days.

Out-of-the-Money Call Options

MRPE

Maturity	<=45 days	45-90 days	>90 days
BSM	2.5070	1.3681	0.8426
Heston (lnsqnonlin)	0.3748	0.0751	0.1461

Implied Volatility

Maturity	<=45 days	45-90 days	>90 days
Market	12.15%	11.48%	11.78%
Heston (lnsqnonlin)	11.86%	11.33%	10.42%

For out-of-the money options, Black Scholes model performs poorly as can be seen from the high MRPE values. Heston performssignificantly better than the Black Scholes model for all moneyness and time-to-maturity combinations.

At-the-Money Call Options

MRPE

Maturity	<=45 days	45-90 days	>90 days
BSM	0.1755	0.2507	0.2684
Heston (lnsqnonlin)	0.2561	0.1162	0.1155

Implied Volatility

Maturity	<=45 days	45-90 days	>90 days
Market	15.94%	13.90%	13.53%
Heston (lnsqnonlin)	11.62%	12.20%	11.89%

For at-the-money options, Heston model outperforms Black Scholes model for middle term and long term. As the time-to-maturity increases, Heston model gives better estimates.

Deep In-the-Money Call Options

MRPE

Maturity	<=45 days	45-90 days	>90 days
BSM	0.0060	0.0061	0.0244
Heston (lnsqnonlin)	0.0133	0.0322	0.0148

Implied Volatility

Maturity	<=45 days	45-90 days	>90 days
Market	15.47%	16.58%	15.83%
Heston (lnsqnonlin)	12.55%	13.22%	15.03%

Like ITM options, Black Scholes model performs better than Heston Model for short-term and middle- term DITM options. But as maturity increases to 90 days, Heston Model starts outperforming Black Scholes Model.

Deep Out-of-the-Money Call Options

MRPE

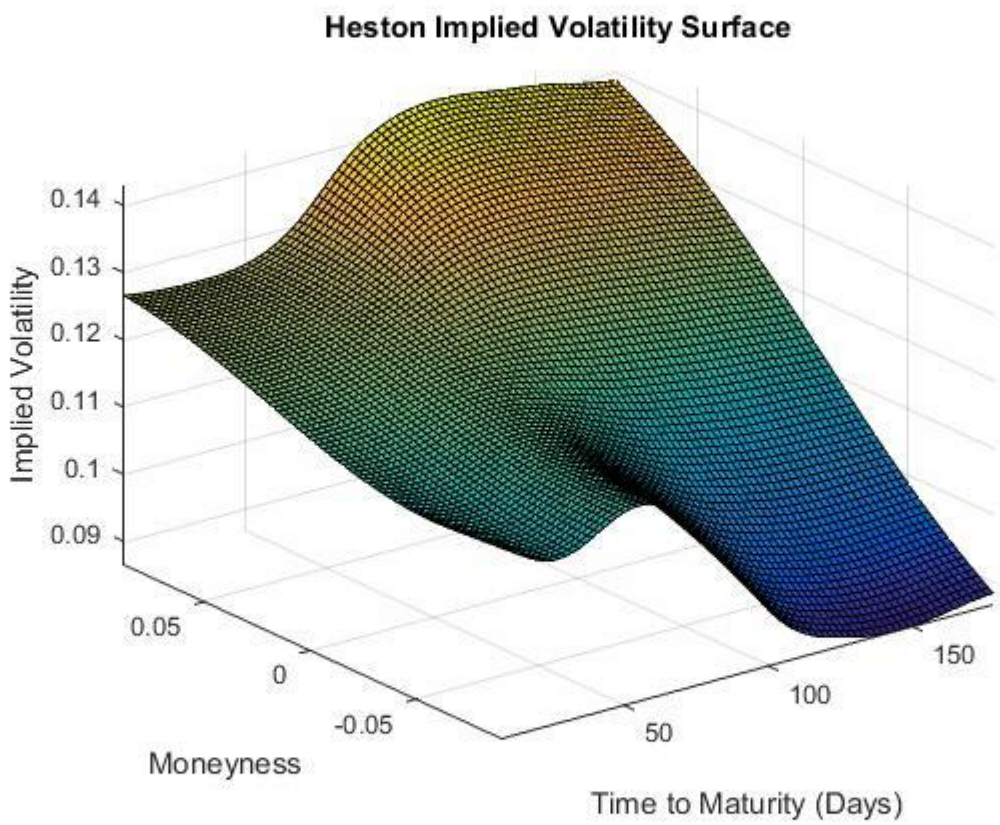
Maturity	<=45 days	45-90 days	>90 days
BSM	10.4604	7.8657	3.9206
Heston (lnsqnonlin)	3.6306	0.7989	0.4783

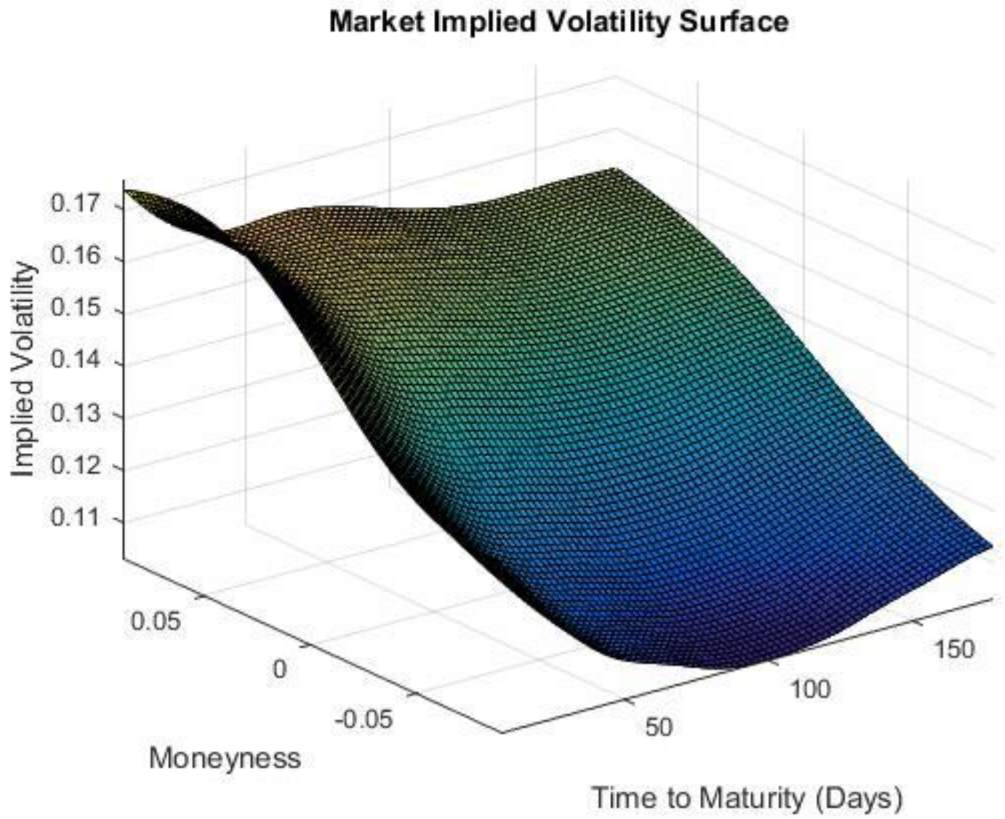
Implied Volatility

Maturity	<=45 days	45-90 days	>90 days
Market	11.95%	10.01%	10.01%
Heston (lnsqnonlin)	13.49%	11.12%	8.16%

Both models perform very poorly for short term DOTM options. But Heston Model significantly outperforms Black Scholes model for all maturities.

Volatility Surfaces





From the volatility surface, it is evident that for short term ITM and DITM options, market implied volatility is quite higher than Heston implied volatility. As a result, Black Scholes model gives better estimates for short term ITM and DITM options. On an average, the implied volatility from Heston model is lower than market implied volatility for different combinations of moneyness and time-to-maturity.

6. CONCLUSION

We observe that Heston model outperforms Black Scholes model in many combinations of moneyness and time-to-maturity combinations. For short term ITM, DITM and ATM options, Black Scholes performs better because Heston model is not able to capture the high implied volatility. But, Heston Model provides better estimates in case of ITM, DITM and ATM options as time-to-maturity increases. For OTMand DOTM options, Heston model gives significantly better estimates than Black Scholes model. In fact, Black Scholes performs very poorly in case of OTM and DOTM options, given the high error values. So, Black Scholes model can be used for short term ITM, DITM and ATM options, while Heston model can be used for other combinations of moneyness and time-to-maturity combinations.

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8. APPENDIX

MATLAB code for Heston Characteristic Function

```
function f=heston_chfun(x,v,t,r,a,u,b,rho,sigma,phi)
h=b-rho.*sigma.*phi.*1i;
d=sqrt(h.^2-(sigma.^2).*(2.*u.*phi.*1i-phi.^2));
g=(h+d)./(h-d);
D=(h+d)./(sigma.^2).*(1-exp(d.*t))./(1-g.*exp(d.*t));
k=(1-g.*exp(d.*t))./(1-g);
C=r.*phi.*1i.*t+a./(sigma.^2).*((h+d).*t-2.*log(k));
f=exp(C+D.*v+1i.*phi.*x);
```

MATLAB code for Heston Price

```
function C=heston_price(St,K,r,t,vt,theta,kappa,sigma,rho)
integrand1=@(phi,St,K,r,t,vt,theta,kappa,sigma,rho)real(exp(-
1i.*phi.*log(K)).*heston_chfun(log(St),vt,t,r,kappa*theta,0.5,kappa
-rho*sigma,rho,sigma,phi)./(1i.*phi));
P1=0.5+(1/pi)*integral(@(phi)integrand1(phi,St,K,r,t,vt,theta,kappa
,sigma,rho),0,100);
integrand2=@(phi,St,K,r,t,vt,theta,kappa,sigma,rho)real(exp(-
1i.*phi.*log(K)).*heston_chfun(log(St),vt,t,r,kappa*theta,-
0.5,kappa,rho,sigma,phi)./(1i.*phi));
P2=0.5+(1/pi)*integral(@(phi)integrand2(phi,St,K,r,t,vt,theta,kappa
,sigma,rho),0,100);
C = St*P1-K*exp(-r*t)*P2;
```

MATLAB code for Local Optimization

```
load data1.txt
x0 = [.01,.01,0.5,-0.9,5];
lb = [0, 0, 0, -1, 0];
ub = [1, 1, 1, 1, 10];
x = lsqnonlin(@costfunction,x0,lb,ub);
heston_sol=[x(1),x(2),x(3),x(4),(x(5)+x(3)^2)/(2*x(2))]
x
minimum = totalcost
```

MATLAB Code for Cost Function

```
function [cost] = costfucntion(x)
fori=1:length(data1)
cost(i)= data1(i,5)-
heston_price(data1(i,1),data1(i,2),data1(i,3),data1(i,4),x(1),x(2),
(x(5)+x(3)^2)/(2*x(2)),x(3), x(4));
end
totalcost=sum(cost)^2
end
```

MATLAB Code for Calculating Out-of-Sample Heston Prices

```
load data2.txt
fori=1:length(data2)
    z(i)=heston_price(data2(i,1),data2(i,2),data2(i,3),data2(i,4),
0.001193, 0.015575, 168.119125, 0.501032, -0.9723);
end
```

MATLAB Code for Black Scholes Price

```
function c=bsm_price(St,K,r,t,sigma)
d1=(log(St./K)+(r+0.5.*sigma.^2).*t)./(sigma.*sqrt(t));
d2=d1-sigma.*sqrt(t);
c=normcdf(d1)*St-normcdf(d2)*exp(-r*t)*K;
```

MATLAB Code for Calculating Out-of-Sample Black Scholes Prices

```
load data2.txt
fori=1:length(data2)
    y(i)=bsm_price(data2(i,1),data2(i,2),data2(i,3),data2(i,4),
0.1706);
end
```

MATLAB code for plotting volatility surface

```
load volsurfacedata.txt
fori=1:length(volsurfacedata)
    time(i)=volsurfacedata(i,1);
    moneyness(i)=volsurfacedata(i,2);
implied_vol(i)=volsurfacedata(i,3);
end

hTime=median(abs(time-median(time)));
surface.hTime=hTime;
hMoneyness=median(abs(moneyness-median(moneyness)));
surface.hMoneyness=hMoneyness;
sorted_time= sort(time);
sorted_moneyness = sort(moneyness);
NT = histcounts(time,sorted_time);
NM = histcounts(moneyness,sorted_moneyness);
NT(NT==0)  = [];
NM(NM==0)  = [];
N=length(NM);
kernel=@(z)exp(-z.*z/2)/sqrt(2*pi);
surface.T=linspace(min(time),max(time),N);
surface.M=linspace(min(moneyness),max(moneyness),N);
surface.IV=nan(1,N);
fori=1:N
for j=1:N
    z=kernel((surface.T(j)-time)/hTime).*kernel((surface.M(i)-
moneyness)/hMoneyness);
surface.IV(i,j)=sum(z.*implied_vol)/sum(z);
end
```

```
end
surf(surface.T,surface.M,surface.IV)
axis tight; grid on;
title('Implied Volatility Surface');
xlabel('Time to Maturity (Days)');
ylabel('Moneyness');
zlabel('Implied Volatility');
```