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Abstract:

The classical newsvendor problem decides the optimal order quantity for a single period, with the assumptions that the selling price and the end of period salvage value are fixed. However, the salvage value or clearance price in many instances depends on the leftover inventory. A fixed salvage value assumption could lead to suboptimal decisions in many situations. We determine the optimal pricing and ordering decision for a newsvendor with variable salvage value. Both additive and multiplicative demand models are considered, and we provide the necessary and sufficient conditions for unique pricing and ordering policies in both cases. We mathematically compare the results against the fixed salvage value newsvendor model and prove that the variable salvage value newsvendor model improves the ordering decision and profit level.

Keywords: inventory, pricing, perishable items, disposal policy, newsvendor, price setting, variable salvage value.

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1. Introduction

Retailers of short product life cycle items, like apparel and fast-moving consumer goods, face the challenges of uncertainty and obsolescence. The retailer stocks up products at the beginning of the selling period, with little information about what the demand would be. If the quantity procured turns out to be different from the actual demand observed, retailer faces either leftovers or stock-outs at the end of the selling period. In the case of over-stocking, it is a common practice among retailer(s) to liquidate the excess inventory through a clearance sale immediately at the end of the selling period [1, 14]. Fisher and Raman [3] report that markdowns in the US have increased from 8% in 1970 to nearly 30% at the turn of the century as a result of accumulating excess inventory, indicating a clear trend in over ordering. The Boxing Day (26th December) of 2012 posted record online clearance sales across UK retail websites¹. Usage of the famous newsvendor model, which assumes a fixed salvage value for determining the optimal order quantity for a single period, in a situation where the salvage value varies with the leftover inventory could lead to sub-optimal solution for the retailer.

We consider a newsvendor model with variable salvage value, such that the leftover inventory clearing price is a function of the leftover quantity itself. This paper addresses the combined price and quantity decisions of a profit maximizing newsvendor when the salvage value is variable. We adopt the price-setting newsvendor paradigm [10] for our analysis. For both additive and multiplicative demand scenario, we provide the sufficient condition for the uniqueness and existence of optimal pricing and ordering decisions. We compare the results against the classical newsvendor model, where the fixed salvage value is obtained by employing the Weighted Average Salvage Value (WASV) heuristic, as proposed by Cachon and K ok [1].

In spite of identifying that salvage value is not constant, Hertz and Schaffir [4] argued that a fixed salvage price is a sufficient approximation. Many scholars [7, 11] have indicated that the salvage value is variable and is dependent on the leftover inventory. There are innumerable examples, ranging from clearance sales of perishable goods to end of season sales of fashion goods, where the price elasticity of demand during clearance sale is not constant [2, 12]. In a few recent studies scholars [1, 14] have incorporated clearance value decision in the stocking

¹ Peachey, K. (December 27, 2012) Sales shoppers set online 'Boxing Day record', BBC News, Retrieved from: <http://www.bbc.com/news/business-20850299>

decision by employing newsvendor framework. These works have assumed the clearance period demand function to be iso-elastic in nature. Cachon and Kök [1] provide four heuristics to estimate the salvage value; among them the WASV heuristic is closest to the optimal. Wang and Webster [14] assume the clearance pricing to be endogenous to the model and then compares between two different types of the contracts based on quantity and price markdown.

Linear demand function often faces the criticism of being restrictive in terms of maximum permissible price [5, 13]; however, the clearance period value or salvage value will always have an upper limit given by the normal selling period price [1, 14]. Hence, linear demand function can realistically model clearance sale demand when the selling season demand is of linear nature. Price-dependency of elasticity according to various stages of product life-cycle is also supported by the behavioural research and industry experts [8]. The iso-elastic demand curve has constant elasticity to demand and when the price approaches zero, the demand approaches infinity [5]. If the selling season demand is iso-elastic, during the clearance sale this characteristic of demand is assumed to be retained [16]. In this paper, we model variable salvage value newsvendor for both linear and multiplicative demand scenario.

The remainder of the paper is organized as follows. Section 2 describes the newsvendor model with different demand scenarios for variable salvage value and optimality conditions. In section 3 we compare the results of our model with the WASV heuristic model.

2. Price-Setting Newsvendor Model with Variable Salvage Value

A price-setting newsvendor firm sells a short life cycle product over a finite selling period that is divided into two parts: normal selling season (T_1) and clearance sale period (T_2). At the beginning of T_1 the firm stocks q units of the product at a per unit cost c . Unit selling price during the selling season is designated by p such that $p > c$. The randomness in demand is price independent and can be modelled by either additive or multiplicative fashion. Following Mills [9] the realised demand is given by, $d(p, \varepsilon) = y(p) + \varepsilon$ where $y(p) = a - bp$ ($a > 0, b > 0$), in the additive case and following Karlin and Carr [6], the realised demand assumes the form, $d(p, \varepsilon) = y(p).\varepsilon$, where $y(p) = ap^{-b}$ ($a > 0, b > 1$), in multiplicative demand scenario. In both the cases $y(p)$ designates the price-dependent part of the demand and ε denotes the random part of the demand defined over the range $[0, \alpha]$. The probability distribution and cumulative

distribution of ε are represented by $f(\cdot)$ and $F(\cdot)$ respectively. We also assume that $f(\cdot)$ and $F(\cdot)$ are differentiable over the entire range $[0, \alpha]$; $F(\cdot)$ is strictly increasing; the boundary conditions of the distribution are: $F(0) = 0$ and $F(\alpha) = 1$. We further define μ as the mean of ε . For the sake of simplicity we assume that, there is no other loss other than the profit loss on every unit of under-stocking.

At the end of the normal selling season, T_1 , the leftover inventory, I , can be expressed as $I = [q - d(p, \varepsilon)]^+$. The leftover items are then sold in the clearance sale period T_2 at a variable unit salvage value, v , and is expressed as $v = v(I)$. Selling season revenue is expressed by $R_1(p, q)$ and clearance sale revenue function alternatively referred as salvage revenue is given by, $R_s = v(I)I$. The profit function of the newsvendor firm is expressed by the equation: $\pi(\cdot) = -cq + R_1(\cdot) + R_s(\cdot)$.

2.1. Additive Demand Scenario

In order to cater to price independent random demand of the selling season, the newsvendor firm orders an excess quantity z above the deterministic part of the demand $y(p)$. Therefore the order quantity (q) is defined as, $q = y(p) + z$. Therefore the leftover quantity can be alternatively expressed by the following equation: $I = [q - d(p, \varepsilon)]^+ = (z - \varepsilon)^+$.

The salvage value is assumed to be a linear function of the left over inventory and is given by, $v(I) = a_v - b_v I$ ($0 \leq a_v \leq p, 0 < b_v$). The salvage revenue function is, $R_s = v(I)I = a_v I - b_v I^2$. First order condition reveals that R_s is maximized at an inventory level given by, $\hat{I} = a_v / 2b_v$. We define, $\delta = z - \hat{I}$; then using these definitions the normal selling season demand, revenue, leftover inventory, clearance sale revenue are expressed in Table 1.

Table 1: Leftover inventory and clearance sale under different demand scenarios

Stochastic demand part in selling period T_1 (u)	Leftover Inventory (I)	Clearance sale volume	Volume to be disposed off at zero salvage value
$0 \leq u \leq \delta$	$I = z - u \geq \hat{I}$	\hat{I}	$I - \hat{I}$
$\delta \leq u \leq z$	$0 \leq I \leq \hat{I}$	I	-

$u > z$	-	-	-
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The selling season expected revenue function, $E[R_1(z, p)]$, and the clearance sale expected salvage revenue, $E[R_s(z)]$, are defined by the following equations,

$$E[R_1(z, p)] = \int_0^z p\{y(p) + u\}f(u)du + \int_z^\alpha p\{y(p) + z\}f(u)du = p\{y(p) + z\} - p \int_0^z F(u)du \quad \dots (1)$$

$$E[R_s(a_v, b_v, z)] = \int_0^z \{a_v - b_v(z - u)\}(z - u)f(u)du = (a_v - 2b_v z) \int_0^z F(u)du + 2b_v \int_0^z uF(u)du$$

... for $z \in [0, \hat{I}]$... (2a)

$$= v_{\min} \hat{I} \int_0^\delta f(u)du + \int_\delta^z \{a_v - b_v(z - u)\}(z - u)f(u)du = (a_v - 2b_v z) \int_\delta^z F(u)du + 2b_v \int_\delta^z uF(u)du$$

... for $z \in (\hat{I}, \alpha]$... (2b)

where v_{\min} is defined as, $v_{\min} = a_v - b_v \hat{I} = a_v / 2$. From the symmetry of the expression, the clearance sale expected salvage revenue, $E[R_s(z)]$, can be alternatively represented by the following equation.

$$E[R_s(a_v, b_v, z)] = (a_v - 2b_v z) \int_\lambda^z F(u)du + 2b_v \int_\lambda^z uF(u)du \quad \dots (3)$$

where, $\lambda = \max(0, \delta)$. Defining $\Theta(z) = \int_z^\alpha (u - z)f(u)du$ and $\Lambda(z) = \int_0^z (z - u)f(u)du$, the overall

expected profit function for the newsvendor is defined as:

$$E[\pi(z, p)] = (p - c)\{y(p) + \mu\} - \{(p - c)\Theta(z) + c\Lambda(z)\} + E[R_s(z)] \quad \dots (4)$$

In equation (4), $(p - c)\{y(p) + \mu\}$ denotes the riskless part of the profit function. The objective is to maximize the expected profit: *Maximize* $E[\pi(z, p)]$. First and second partial derivatives of the

profit function taken with respect to p are given as follows:

$$\partial E[\pi(z, p)] / \partial p = 2b(p^0 - p) - \Theta(z) \quad \dots (5)$$

$$\partial^2 E[\pi(z, p)]/\partial p^2 = -2b < 0 \quad \dots (6)$$

where $p^0 = \frac{a+bc+\mu}{2b}$. The expression p^0 denotes the optimal riskless price and it maximizes the riskless part of the profit function: $(p-c)\{y(p)+\mu\}$. The expression obtained for riskless price is exactly similar to the one obtained by Petruzzi and Dada [10]. From the second order derivatives of the expected profit function with respect to z , it can be shown that $E[\pi(z, p)]$ is concave in z for a given value of p over both the ranges $[0, \hat{I}]$ and $(\hat{I}, \alpha]$. Therefore we can adopt Zabel's method [15] of optimization for the expected profit function: first optimize p for a given z and subsequently search over the resulting optimal trajectory in order to maximize $E[\pi(z, p^*)]$. From equation (5) the optimal price (p^*) can be expressed as a function of z and is expressed by the following equation.

$$p^* = p(z) = p^0 - \{\Theta(z)/2b\} \quad \dots (7)$$

Substituting $p^* = p(z)$ in $E[\pi(z, p)]$ the optimization problem gets converted into a single variable maximization. Theorem 1 demonstrates how optimal z^* can be computed.

Theorem 1. For variable salvage value: $v(I) = a_v - b_v I$, the optimal decisions of a newsvendor for a single period are given as follows: optimal order quantity is defined by : $q^* = d(p^*) + z^*$. Optimal price is defined by: $p^* = p^0 - \frac{\Theta(z^*)}{2b}$ where $p^0 = \frac{a+bc+\mu}{2b}$. Optimal z^* is determined according to the following:

(i) $E[\pi\{z, p(z)\}]$ is concave over the entire range $[0, \alpha]$ and therefore an exhaustive search over the range will yield the optimal z^* .

(ii) If $(p - a_v)\{1 - F(\hat{I})\} + (a_v - c) \leq 0$ then, $z^* \in [0, \hat{I}]$ and otherwise $z^* \in (\hat{I}, \alpha]$ where $\hat{I} = a_v/2b_v$.

(iii) The optimal z^* satisfies the condition, $(p - c) - (p - a_v)F(z) - 2b_v \int_{\lambda}^z F(u)du = 0$.

Proof. See the appendix.

Theorem 1 identifies the conditions for which the optimal solution for a single period variable salvage value newsvendor problem can be identified analytically. In the following section we analyze the optimal condition(s) for multiplicative demand.

2.2. Multiplicative Demand Scenario

In the case of multiplicative demand the order quantity (q) is defined as, $q = d(p).z$ where $y(p) = ap^{-b}$ ($a > 0, b > 1$). The leftover quantity is expressed by the following equation: $I = [q - d(p, \varepsilon)]^+ = (z - \varepsilon)^+ .d(p)$. The salvage value is assumed to be a multiplicative function of the left over inventory and is given by, $v(I) = a_v I^{-b_v}$ ($0 < b_v < 1$). In order to obtain closed form solutions, we further assume, $b_v = \frac{1}{b}$ without the loss of generality. In order to have salvage value ($v(I)$) to be less than the selling season price (p) the following condition is to be satisfied: $a_v \leq (a\alpha)^{b_v}$. The salvage revenue function is, $R_s = v(I)I = a_v I^{1-b_v}$. Unlike additive demand scenario, R_s does not maximize at a particular leftover inventory level (I). The selling season expected revenue function, $E[R_1(z, p)]$, and the clearance sale expected salvage revenue, $E[R_s(z)]$, are given by the following equations,

$$E[R_1(z, p)] = \int_0^z p\{d(p)u\}f(u)du + \int_z^\alpha p\{d(p).z\}f(u)du = py(p)\mu - py(p)\Theta(z) \quad \dots (8)$$

$$E[R_s(a_v, b_v, z)] = \int_0^z a_v \{(z-u)y(p)\}^{1-b_v} f(u)du = a_v \{y(p)\}^{1-b_v} \int_0^z (z-u)^{1-b_v} f(u)du = a_v \{y(p)\}^{1-b_v} \Psi(z) \quad \dots (9)$$

where, $\int_0^z (z-u)^{1-b_v} f(u)du = \Psi(z)$. $\Theta(z)$ is previously defined in section 2.1. For variable salvage

value, the expected profit function of the newsvendor, $E[\pi(z, p)]$, is given by,

$$E[\pi(z, p)] = D(p) - [(p - c)\Theta(z) + c\Lambda(z)]y(p) + E[R_s(a_v, b_v, z)] \quad \dots (10)$$

where, $D(p) = (p - c)y(p)\mu$ and $\Lambda(z)$ is previously defined. In multiplicative demand case, p^0 is defined as the price level that maximizes $D(p)$. By taking first and second derivatives of $D(p)$ with respect to p we obtain:

$$\frac{d}{dp}\{D(p)\} = (1-b)\mu \frac{y(p)}{p} \left[p - \frac{bc}{b-1} \right] = 0, \text{ at } p = \frac{bc}{b-1} \quad \dots (11)$$

$$\frac{d^2}{dp^2}\{D(p)\} = (1-b)\mu \frac{y(p)}{p} \left[1 - \frac{(1+b)}{p} \left(p - \frac{bc}{b-1} \right) \right] = (1-b)\mu \frac{y(p)}{p} < 0, \text{ at } p = \frac{bc}{b-1} \quad \dots (12)$$

It is evident from equation (11) and (12) that $D(p)$ maximizes at $p^0 = \frac{bc}{b-1}$ and this result conforms to that obtained by Petruzzi and Dada [10]. The optimal price at which the expected profit function is maximized is given by Lemma 1.

Lemma 1. *For a given value of z , the optimal price, p^* , is expressed as a unique function of z :*

$$p^* = \frac{(b-1)\mu p^0 + bc\{\Lambda(z) - \Theta(z)\}}{(b-1)\{\mu - \Theta(z) - a_v a^{-b_v}\}} \text{ where } p^0 = \frac{bc}{b-1}$$

Proof. See the appendix.

Subsequently to maximize $E[\pi(z, p)]$ we follow the same sequential procedure as detailed in section 2.1. Lemma 1 establishes optimal price as a function of z , $p^* = p(z)$, and this price is substituted back into the expected profit function. Thus, the maximization problem is reduced over a single variable. The optimal z^* is computed in accordance with Theorem 2.

Theorem 2. *For variable salvage value the optimal decisions of a newsvendor for a single period are given as follows: optimal order quantity is defined by: $q^* = d(p^*).z^*$. Optimal price is specified by Lemma 1 and optimal z^* is determined by solving the equation,*

$$(p - c)\{1 - F(z)\} - cF(z) + a_v a^{-b_v} p(z) \frac{d\{\Psi(z)\}}{dz} = 0$$

Proof. See the appendix.

In the following section we compare the optimal solution obtained for a variable salvage value newsvendor model with that obtained for the classical newsvendor to understand the

conditions under which the former performs superiorly. For the purpose of brevity we restrict the comparison to the additive demand case.

3. Comparison with the Classical Newsvendor Model

As previously mentioned in section 1, Cachon and K ok [1] have proposed different heuristics for estimating the fixed salvage value. They have demonstrated that the weighted average salvage value (WASV) heuristic provides the solution that is closest to the optimal one. The WASV heuristic computes the fixed salvage value by the relationship $v(q_C) = R_s(q_C)/I(q_C)$, where q_C represents the classical newsvendor optimal order quantity. At the classical newsvendor optimal order quantity, $v(q_C) = p - (p - c)/F(q_C)$. In this section, we evaluate the variable salvage value newsvendor model against the classical newsvendor model where salvage value is computed using the WASV heuristic.

Lemma 2. *The optimal profit in the variable salvage value newsvendor model will at least be equal to the optimal profit obtained for the classical newsvendor in the case of additive demand.*

Proof. For additive demand, the fixed salvage value as per WASV heuristic is represented by $v(q_C) = R_s(q_C)/I(q_C)$ where $q_C = y(p) + z_C$. $y(p)$ is the deterministic part of the demand and z_C is the excess order quantity computed according to classical newsvendor model. The classical newsvendor profit function (π_C) and variable salvage value newsvendor profit function (π_{VSP}) are given by,

$$\pi_C(p, v, q) = E[R_1(p, q)] + E[R_s(v, q)] - cq \quad \dots (13)$$

$$\pi_{VSP}(p, a_v, b_v, q) = E[R_1(p, q)] + E[R_s(a_v, b_v, q)] - cq \quad \dots (14)$$

For a given price p , at $q = q_C$ the clearance revenue yield through both the models would be same, $E[R_s(v, q_C)] = E[R_s(a_v, b_v, q_C)]$ when the clearance sale demand is given by $v(I) = a_v - b_v I$, and the clearance price for classical newsvendor model is determined by WASV heuristic. Therefore, at $q = q_C$, $\pi_{VSP}(p, a_v, b_v, q_C) = \pi_C(p, v, q_C)$. In other words, the optimal profit for variable salvage value newsvendor model will at least be equal to the classical newsvendor optimal profit.

We subsequently prove in Lemma 3 that variable salvage value model also results in improved quantity decision compared to a classical newsvendor model.

Lemma3: *If the stochastic component of demand follows the relationship $\mu + SD(\zeta) < 1$ in the case of additive demand, then the optimal order quantity, q^* , for a variable salvage value newsvendor is lesser than the classical newsvendor optimal order quantity, q_c , where $\zeta = \varepsilon/z^*$ represents a truncated distribution of ε over $[0, z^*]$.*

Proof: See the appendix.

Through Lemma 2 we prove that the newsvendor model with fixed salvage value results in profit loss compared to a newsvendor model where salvage value is assumed to be a function of leftover inventory. Through Lemma 3 we establish the condition under which the newsvendor model with fixed salvage value results in over ordering. This condition holds true for commonly used demand distributions like normal, gamma and uniform distributions. The newsvendor model results are extensively used in a wide variety of industry contexts including many where the salvage value cannot be assumed to be fixed. The results presented in this paper provide a simple yet realistic approach for incorporating variable salvage value.

APPENDIX.

Proof of Theorem 1. From the first and second order derivative of $E[\pi(z, p)]$ we establish the convexity of the profit function over the entire range $[0, \alpha]$.

$$dE[\pi(z, p)]/dz = (p - c) - (p - a_v)F(z) - 2b_v \int_{\lambda}^z F(u)du \quad \dots (A1)$$

$$\frac{d^2 E[\pi(z, p)]}{dz^2} = -\{1 - F(z)\} \left\{ (p - a_v)r(z) + 2b_v \frac{F(z) - F(\lambda)}{1 - F(z)} - \frac{1 - F(z)}{2b} \right\} \quad \dots (A2)$$

From (A2) it is evident that, $E[\pi(z, p)]$ is convex over the entire range $[0, \alpha]$. when z is IGFR distributed and also satisfies the condition $(p - a_v)r(z) + 2b_v \frac{F(z) - F(\lambda)}{1 - F(z)} > \frac{1 - F(z)}{2b}$. The optimal z value can be investigated by looking at the zeroes of the function $\Delta(z) = dE[\pi(z, p)]/dz$. The first-order and second-order derivatives of the function $\Delta(z)$ are given as follows:

$$\frac{d\Delta(z)}{dz} = -\{1 - F(z)\} \left\{ (p - a_v)r(z) + 2b_v \frac{F(z) - F(\lambda)}{1 - F(z)} - \frac{1 - F(z)}{2b} \right\} \quad \dots (A3)$$

$$\left. \frac{d^2 \Delta(z)}{dz^2} \right|_{d\Delta(z)/dz=0} = -\{1 - F(z)\} \left\{ (p - a_v) \frac{d\{r(z)\}}{dz} + \frac{f(z)}{2b} + 2b_v \frac{r(z)[1 - F(\lambda)] - f(\lambda)}{1 - F(z)} \right\} \quad \dots (A4)$$

We have, $\frac{d\Delta(z)}{dz} < 0$ from the profit function derivation. $\Delta(z)$ is monotone decreasing over the range $[0, \alpha]$ iff $(p - a_v) \frac{d\{r(z)\}}{dz} + \frac{f(z)}{2b} + 2b_v \frac{r(z)[1 - F(\lambda)] - f(\lambda)}{1 - F(z)} > 0$. The value of $\Delta(z)$ at

the end points of the range $[0, \hat{I}]$ are given by,

$$\Delta(0) = p - c > 0$$

$$\Delta(\hat{I}) = (p - c) - (p - a_v)F(\hat{I}) - 2b_v \int_0^{\hat{I}} F(u)du$$

From the properties of integral, $(p - a_v)\{1 - F(\hat{I})\} + (a_v - c) \geq \Delta(\hat{I}) \geq (p - a_v)\{1 - F(\hat{I})\} - c$.

Under the condition $(p - a_v)\{1 - F(\hat{I})\} + (a_v - c) \leq 0$, $\Delta(0)$ and $\Delta(\hat{I})$ are opposite in sign. Then the optimal z^* lies in the range $[0, \hat{I}]$ and it satisfies the condition,

$$(p - c) - (p - a_v)F(z) - 2b_v \int_0^z F(u)du = 0 \quad \dots (A5)$$

The value of $\Delta(z)$ at the end points of the range $(\hat{I}, \alpha]$ are given by,

$$\Delta(\hat{I}+) = (p - c) - (p - a_v)F(\hat{I}) - 2b_v \int_{\delta}^{\hat{I}} F(u)du$$

$$\Delta(\alpha) = (a_v - c) - 2b_v \int_{\alpha - \hat{I}}^{\alpha} F(u)du$$

From the properties of integral, $(p - a_v)\{1 - F(\hat{I})\} + (a_v - c) \geq \Delta(\hat{I}+) \geq (p - a_v)\{1 - F(\hat{I})\} - c$

The search of optimal z^* extends into $(\hat{I}, \alpha]$ iff $(p - a_v)\{1 - F(\hat{I})\} + (a_v - c) > 0$. Under the condition $a_v\{1 - F(\hat{I})\} - c < 0$, we have $\Delta(\alpha) < 0$. Therefore $\Delta(\alpha)$ and $\Delta(\hat{I})$ are opposite in sign. Then the optimal z^* lies in the range $(\hat{I}, \alpha]$ and it satisfies the condition,

$$(p - c) - (p - a_v)F(z) - 2b_v \int_{\delta}^z F(u)du = 0 \quad \dots (A6)$$

Combining (A5) and (A6) we get, $(p - c) - (p - a_v)F(z) - 2b_v \int_{\lambda}^z F(u)du = 0$.

Proof of Lemma 1. The first order and second order partial derivatives of the expected profit function with respect to p are given by the following equations,

$$\partial E[\pi(z, p)] / \partial p = y(p).B(z, p) / p \quad \dots (A7)$$

$$\frac{\partial^2 E[\pi(z, p)]}{\partial p^2} = \frac{y(p)}{p} \left[\frac{\partial B(z, p)}{\partial p} - \frac{b+1}{p} B(z, p) \right] \quad \dots (A8)$$

where, $B(z, p) = (b - 1)\mu p^0 + bc\{\Lambda(z) - \Theta(z)\} - (b - 1)\{\mu - \Theta(z) - a_v a^{-b_v}\}p$.

The first order partial derivative of $B(z, p)$ with respect to p is given by,

$$\partial B(z, p)/\partial p = -(b-1)\{\mu - \Theta(z) - a_v a^{-b_v}\} < 0$$

Therefore, at $p^* = \frac{(b-1)\mu p^0 + bc\{\Lambda(z) - \Theta(z)\}}{(b-1)\{\mu - \Theta(z) - a_v a^{-b_v}\}}$, $B(z, p) = 0$. From (A7) and (A8) it is evident

that $E[\pi(z, p)]$ is maximum at, $p = p^*$.

Proof of Theorem 2. At $p = p^* \equiv p(z)$ the first order derivative of the expected profit function

$$\text{is given by, } \frac{dE[\pi(z, p(z))]}{dz} = -\left[(p-c)\frac{d\{\Theta(z)\}}{dz} + c\frac{d\{\Lambda(z)\}}{dz} - a_v a^{-b_v} p \frac{d\{\Psi(z)\}}{dz}\right]y(p)$$

The optimal z value can be investigated by looking at the zeroes of the function,

$$X(z) = (p-c)\frac{d\{\Theta(z)\}}{dz} + c\frac{d\{\Lambda(z)\}}{dz} - a_v a^{-b_v} p \frac{d\{\Psi(z)\}}{dz} = -(p-c)\{1-F(z)\} + cF(z) - a_v a^{-b_v} p \frac{d\{\Psi(z)\}}{dz}$$

If $X(0)$ and $X(\alpha)$ are of opposite signs then it can be concluded that optimal z lies within the range $[0, \alpha]$. Therefore the value of $X(\cdot)$ is dependent on the behaviour of the function $\frac{d\{\Psi(z)\}}{dz}$.

Since $\{1 - (u/z)\}^{-b_v}$ is a convergent series, therefore approximating the expression by first three terms of the series the value of the first order derivative of $\Psi(z)$ is approximated as follows,

$$\frac{d\{\Psi(z)\}}{dz} = \frac{(1-b_v)}{z^{b_v}} \int_0^z \left(1 - \frac{u}{z}\right)^{-b_v} f(u) du \approx \frac{(1-b_v)}{z^{b_v}} \int_0^z \left[1 + \frac{b_v u}{z} + \frac{b_v(1+b_v)u^2}{2z^2}\right] f(u) du \quad \dots (A9)$$

Defining, $\zeta = (Y_z/z)$ where Y_z represents a random variable that corresponds to the truncated distribution of u over $[0, z]$, (A9) can be rewritten as follows,

$$\frac{d\{\Psi(z)\}}{dz} \approx \frac{(1-b_v)F(z)}{z^{b_v}} \left[1 + b_v E(\zeta) + \frac{b_v(1+b_v)}{2} E(\zeta^2)\right] \quad \dots (A10)$$

From (A10) it is evident that, $d\{\Psi(z)\}/dz > 0$; for small z values, $d\{\Psi(z)\}/dz \rightarrow 0$ and as z

approaches α , we have $\lim_{z \rightarrow \alpha} \frac{d\{\Psi(z)\}}{dz} \approx \frac{1-b_v}{\alpha^{b_v}} \left[1 + \frac{\mu b_v}{\alpha} + \frac{b_v(1+b_v)}{2} \left\{Var(\zeta) + \left(\frac{\mu}{\alpha}\right)^2\right\}\right] > 0$.

Therefore, as $z \rightarrow 0$ we have, $X(0) \rightarrow \{-(p-c)\} < 0$ and as $z \rightarrow \alpha$, we will have $X(\alpha) > 0$

when, $c > a_v a^{-b_v} p(\alpha) \left[\lim_{z \rightarrow \alpha} \frac{d\{\Psi(z)\}}{dz}\right]$.

Proof of Lemma 3. From Lemma 2 we have, $\pi_{VSP}(p, a_v, b_v, q) - \pi_C(p, v, q) \geq 0$ for a given price (p) and quantity (q). The clearance sale revenue in the classical newsvendor model is equal to $vI(q)$, where $v = R_s(q_C)/I(q_C)$ using WASV heuristic. Hence, the difference between the variable salvage value newsvendor profit and the classical newsvendor profit is given by,

$$\Delta\pi = \pi_{VSP}(p, a_v, b_v, q) - \pi_C(p, v, q) = R_s(a_v, b_v, q) - vI(q) = I(q) \left\{ \frac{R_s(\cdot)}{I(q)} - \frac{R_s(q_C)}{I(q_C)} \right\} \quad \dots (A11)$$

From (A11) it is evident that the difference in profit, $\Delta\pi$, would be non-negative for a positive order quantity $q(\leq q_C)$ iff $\frac{R_s(\cdot)}{I(q)}$ is decreasing in q . For additive demand the ratio of clearance sale revenue to leftover inventory is presented by the following equation.

$$\frac{R_s(\cdot)}{I(q)} \equiv \frac{R_s(\cdot)}{I(z)} = [a_v - 2b_v \{y(p) + z\}] + 2b_v \cdot \left[\int_0^z uF(u)du \right] / \left[\int_0^z F(u)du \right] \quad \text{for } z \in [0, \hat{I}] \quad \dots (A12)$$

$$= [a_v - 2b_v \{y(p) + z\}] \frac{\int_0^z F(u)du}{\int_0^z F(u)du} + 2b_v \frac{\int_0^z uF(u)du}{\int_0^z F(u)du} \quad \text{for } z \in [\hat{I}, \alpha] \quad \dots (A13)$$

The first order partial derivative of the ratio with respect z to is given by,

$$\frac{\partial}{\partial z} \left\{ \frac{R_s(\cdot)}{I(z)} \right\} = \frac{2b_v}{\left\{ \int_0^z F(u)du \right\}^2} \left[\left\{ \int_0^z F(u)du \right\} \left\{ \int_0^z uf(u)du \right\} - \left\{ \int_0^z uF(u)du \right\} \left\{ \int_0^z f(u)du \right\} \right] \quad \dots \text{for } z \in [0, \hat{I}] \quad \dots (A14)$$

$$= -2b_v + \frac{2b_v F(z)}{\left\{ \int_0^z F(u)du \right\}^2} \left[\left\{ z \int_0^z F(z)dz + \int_0^\delta uF(u)du \right\} - \left\{ (q-s) \int_0^\delta F(u)du + \int_0^z uF(u)du \right\} \right] \quad \dots \text{for } z \in [\hat{I}, \alpha] \quad \dots (A15)$$

By changing the order of integration, for $z \in [0, \hat{I}]$, (A14) can be rewritten as:

$$\frac{\partial}{\partial z} \left\{ \frac{R_s(\cdot)}{I(z)} \right\} = \frac{2b_v \{F(z)\}^2}{\left\{ \int_0^z F(u) du \right\}^2} \left[A_1 A_2 - \frac{1}{2} A_3 \right]$$

where $A_1 = \int_0^z u f(u) du / F(z)$, $A_2 = \int_0^z (z-u) f(u) du / F(z)$ and $A_3 = \int_0^z (z^2 - u^2) f(u) du / F(z)$. If

Y_z represents a random variable that corresponds to the truncated distribution of u over $[0, z]$, then by defining $\zeta = (Y_z/z)$ we have, $A_1 = zE(\zeta)$, $A_2 = zE(1-\zeta)$ and $A_3 = z^2E(1-\zeta^2)$. $R_s(\cdot)/I(z)$ reduces in z over the range $[0, \hat{I}]$ for $A_1 A_2 < A_3/2$. Upon simplification $A_1 A_2 < A_3/2$ condition yields, $E(\zeta) + SD(\zeta) < 1$, where $SD(\zeta)$ represents the standard deviation of the distribution ζ .

If Y_δ represents a random variable that corresponds to the truncated distribution of u over $[0, \delta]$ and $\psi = (Y_\delta / \delta)$ then over the range $[\hat{I}, \alpha]$, (A15) can be represented as follows,

$$\frac{\partial}{\partial z} \left\{ \frac{R_s(\cdot)}{I(z)} \right\} = - \frac{2b_v}{\{1 - E(\zeta)\}^2} \left[\left[\{1 - E(\zeta)\}^2 - \left\{ \frac{1}{2} - E(\zeta) + \frac{1}{2} E(\zeta^2) \right\} \right] + \left(\frac{\delta}{z} \right)^2 \frac{F(\delta)}{F(z)} \left\{ \frac{1}{2} - E(\psi) + \frac{1}{2} E(\psi^2) \right\} \right]$$

From this expression it is evident that, $R_s(\cdot)/I(z)$ reduces in z over the range $[\hat{I}, \alpha]$ for

$\{1 - E(\zeta)\}^2 - \{(1/2) - E(\zeta) + E(\zeta^2)/2\} > 0$. This condition can be alternatively represented as,

$$E(\zeta) + SD(\zeta) < 1.$$

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