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Abstract

For certain types of goods, the multiple unit auctions have to be conducted sequentially. One probable reason for this is that the different units of the goods are not available together for putting up for sale. This might happen when the objects are available in batches to the auctioneer and do not come together. In such sequential auctions, there may be waiting costs involved for the bidders. This note makes an attempt to look at the bidding behaviour when waiting costs are private information for the bidders and therefore constitute types. The results derived suggest that, in a symmetric, independent, private valuations framework, with risk neutral bidders, there exists a symmetric Perfect Bayesian equilibrium bidding strategy, for which the bids are increasing in types in the first stage and decreasing in types in the second stage.

1 Introduction

There are different commodities for which multiple unit auctions have to be conducted sequentially. This might happen when the objects are available in batches to the auctioneer and do not come together. As example we can consider fish or flower auctions. In many places fishes are sold together as a bundle by wholesalers. The retailers who purchase from the wholesalers then come to sell in the local markets. Now, the bundles of fishes may arrive at different points of time. How early a retailer can reach a market might be decisive for how much competition he/she will face in that market. The one who reaches the market earliest, in fact may enjoy a monopoly for some time. The more the time lapses the higher becomes the competition in the market. Thus waiting to enter the market may prove costly for the retailers. A similar situation can be imagined in context of telecommunications spectrum auctions. If the licenses are auctioned sequentially, then the first operator in the market might have advantages in terms of market share over the late entrants. Waiting costs in fish markets can occur in another way. As noted by

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Jensen (2007), the retail markets may be (and are quite likely to be) located at different distances from the wholesale market. The larger the distance of a retail market from a wholesale market, the higher is the cost of commuting to that retail market. The bidders who win first would therefore obviously try to capture the nearest markets, while the bidders who win later would be compelled to locate themselves in further markets. Now, traveling a longer distance not only means a higher commuting cost, the time involved in commuting is likely to result in poorer qualities of fish (since fish is a perishable object). Thus the compulsion to locate in a distant market might inflict substantial costs on the sellers (bidders in the wholesale fish market). Moreover, if they reach the markets very late then the buyers might have departed already (such an instant has been noted by Jensen (2007)). In all the above mentioned situations, the bidders with higher waiting costs are likely to bid higher in the first stage and lower in the second stage. The underlying intuition is quite straightforward. The higher the waiting cost, the lower is the net effective value in the second stage. The objective of this paper is to find out whether such bidding strategies indeed constitute a perfect Bayesian equilibrium for a two stage auction.

If such an equilibrium indeed exists, then the prices will be lower in the second stage. In the standard auction theory, the incidence of *declining price anomaly* in context of sequential auctions has been studied widely. We mention some notable ones among them. Ashenfelter (1989) mentioned this event of declining price anomaly in context of wine auctions. McAfee and Vincent (1993) relate the incidence of declining prices to the nodecreasing absolute risk aversion in the bidders' behaviour. Gale and Hausch (1994) refer to a situation where the seller chooses order of sale and right-to-choose auctions in which the winner chooses her preferred item from the remaining items and show that right-to-choose ensures efficiency and declining prices. Bernhardt and Scoones (1994) consider a case of "stochatically identical" objects to explain the decreasing prices phenomenon in sequential auctions. Branco (1997) explains decreasing prices in sequential auctions in terms of complementarities among the objects. Ginsburgh (1998) shows that the declining price anomaly in wine auctions is due to the fact that most bids are entered by absentee bidders using non-optimal bidding strategies. However, in sequential auctions, whether in one stage bids are increasing in types and in another, they are decreasing in types, have not been looked at before. This paper makes an attempt in this direction. The following section 2 elaborates the model and establishes the results and section 3 concludes the paper.

2 Model

We consider a simple model involving a two-stage auction to start with. In each stage one object is sold using the firs price sealed bid auction format. The assumptions of the model are as follows:

1. There are $n \geqslant 3$ risk neutral bidders with no budget constraints.

2. There are two identical indivisible objects up for sale in two stages, with one unit being sold in each stage.

3. Each bidder has demand for a single unit, so that after the first stage the winning bidder of that stage exits.

4. The value that each bidder attaches to one of the objects is common for everyone, denoted by V and it is common knowledge. Before the beginning of the second stage, the winning bid of the first stage is disclosed.

5. There is waiting cost for each individual bidder i denoted by $c_i$and it is a private information to bidder i.

6. c_i-s are distributed independently and identically over the interval $[\underline{c}, \overline{c}]$ following the same continuous distribution function $F(.)$with density $f(.)$and has full support.

7. Thus in the second stage the value for the remaining object for bidder i is $V - c_i$. We denote this net value by x_i.

8. The distribution function of the x_i-s is denoted by the continuous function$G(.)$, with the corresponding density being $g(.)$, over the interval $[\underline{x}, \overline{x}]$. Here, $\underline{x} = V - \overline{c}$ and $\overline{x} = V - \underline{c}$.

Thus we are in a symmetric independent private valuations framework. As we can make out form the above assumptions, in the first stage the bids will be increasing functions of the waiting costs while in the second stage the bids will be declining in waiting costs. This is because of the fact that, the higher the waiting cost, the less will be the net effective value of the object in the second stage (i.e. the higher will be the reduction in value). Therefore, the bidders with higher waiting costs are likely to bid higher in the first stage. In the second stage auction, a bidder with a higher waiting cost faces less value, and therefore in this stage that bidder will submit a lower bid. In the second stage, there remain $(n-1)$ bidders. In the concerned two-stage auction, waiting cost is c. So the net effective value in the second stage is $V - c$. The distribution function of c is $F(.)$, with a continuous density $f(.)$, over the interval $[\underline{c}, \overline{c}]$. Since $x = V - c$, if C_1 is the highest order statistic for c , then $X_1 = V - C_1$ is the lowest order statistic for x using this transformation, without any loss of generality, we can treat x as the type. Thus if we can prove that in our intended symmetric perfect Bayesian equilibrium for this two stage auction, the bids in the first and second stages are respectively decreasing and increasing in x, we effectively prove that the bids in the first and second stages, the bids are respectively increasing and decreasing in c in the symmetric perfect Bayesian equilibrium.

We start our analysis from the second stage and work backward. We have assumed that the winning bid of the first stage is disclosed before the beginning of the second stage and the winning bidder exits after this stage. Since bids are monotonic functions of types, therefore, disclosure of the winning bid reveals the winning type. Now, the highest type, denoted by c_1, will be the winning type. Since $x = V - c$, therefore, we denote by $x_1 = V - c_1$ as the lowest value of x. The probability of winning in the second stage is therefore conditional on the fact that the winning type in the first stage auction is x_1(Appendix A.1). In a symmetric framework, the expected payoff function of any individual representative bidder, with type x, but bids as though type is z, can be written as

$$\Pi_{II}\left(z;x\right)=\frac{G\left(z\right)^{n-2}}{\left(1-G\left(x_1\right)\right)^{n-2}}\left(x-\beta_{II}\left(z\right)\right)$$

Optimising this, in a symmetric equilibrium we obtain,

$$\beta_{II}\left(x\right)=\frac{1}{G\left(x\right)^{n-2}}\int_{\underline{x}}^{x}yd\left[G\left(y\right)^{n-2}\right]$$

and

$$\Pi_{II}\left(x;x\right)=\frac{1}{\left(1-G\left(x_1\right)\right)^{n-2}}\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy$$

It can be routinely checked that this bidding strategy is increasing in x and it satisfies the second order condition as well.

Next, we consider the first stage of the auction. The expected payoff of the same bidder, whose type is x, but submits a bid such that as though type is z, can be written as

$$\Pi_I\left(z;x\right)=\left(V-\beta_I\left(z\right)\right)\left(1-G\left(z\right)\right)^{n-1}+\left(1-\left(1-G\left(z\right)\right)^{n-1}\right)E\left[\Pi_{II}\left(x;x\right)|X_1<z\right]$$

Here, $\left(1-G\left(z\right)\right)^{n-1}$is the probability of winning in the first stage (Appendix A.1). It follows that the expected payoff for a bidder in the first stage auction also considers the contingency of losing in the first stage and therefore incorporates the expected payoff from the second stage auction conditional on not winning in the first stage. Since the bidder with the lowest type X_1 wins in the first stage auction, for the bidder who loses by bidding as a function of type z, $X_1<z$ must hold true. Here $E\left[\Pi_{II}\left(x;x\right)|X_1<z\right]$is the expected payoff from the second stage conditional on the fact that $X_1<z$.

Now

$$\begin{aligned}
&E\left[\Pi_{II}\left(x;x\right)|X_1<z\right]\\
&=E\left[\tfrac{1}{(1-G(X_1))^{n-2}}\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy\,|X_1<z\right]\\
&=\left\{\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy\right\}E\left[\tfrac{1}{(1-G(X_1))^{n-2}}\,|X_1<z\right]\\
&=\frac{\left\{\int_{\underline{x}}^{x}G(y)^{n-2}dy\right\}}{\left(1-(1-G(z))^{n-1}\right)}\int_{\underline{x}}^{z}\frac{d\left[1-(1-G(X_1))^{n-1}\right]}{(1-G(X_1))^{n-2}}\\
&=\frac{\left\{\int_{\underline{x}}^{x}G(y)^{n-2}dy\right\}}{\left(1-(1-G(z))^{n-1}\right)}\int_{\underline{x}}^{z}\frac{(n-1)(1-G(X_1))^{n-2}g(X_1)dX_1}{(1-G(X_1))^{n-2}}\\
&=\frac{(n-1)\left\{\int_{\underline{x}}^{x}G(y)^{n-2}dy\right\}}{\left(1-(1-G(z))^{n-1}\right)}G\left(z\right)
\end{aligned}$$

Therefore, the expected payoff function now can be written as

$$\Pi_I(z;x) = (V - \beta_I(z))(1 - G(z))^{n-1} + (n-1)\left\{\int_{\underline{x}}^{x} G(y)^{n-2}\,dy\right\} G(z)$$

From the first order conditions of maximisation and in a symmetric equilibrium, we obtain

$$\begin{aligned}
(V - \beta_I(x))\, d\left[(1 - G(x))^{n-1}\right] - \beta_I^{/}(x)(1 - G(x))^{n-1} + (n-1)\left\{\int_{\underline{x}}^{x} G(y)^{n-2}\,dy\right\} g(x) = 0 \\
\Rightarrow d\left[\beta_I(x)(1 - G(x))^{n-1}\right] = V d\left[(1 - G(x))^{n-1}\right] + (n-1)\left\{\int_{\underline{x}}^{x} G(y)^{n-2}\,dy\right\} g(x) \\
\Rightarrow -\beta_I(x)(1 - G(x))^{n-1} = -V(1 - G(x))^{n-1} + (n-1)\int_{x}^{\overline{x}}\left\{\int_{\underline{x}}^{w} G(y)^{n-2}\,dy\right\} g(w)\,dw \\
\Rightarrow \beta_I(x) = V - \frac{(n-1)}{(1-G(x))^{n-1}}\int_{x}^{\overline{x}}\left\{\int_{\underline{x}}^{w} G(y)^{n-2}\,dy\right\} g(w)\,dw
\end{aligned}$$

From this we look at the sign of $\beta_I^{/}(x)$. We obtain

$$\beta_I^{/}(x) = \frac{(n-1)g(x)}{(1-G(x))^{n-1}}\left[\int_{\underline{x}}^{x} G(y)^{n-2}\,dy - \frac{(n-1)}{(1-G(x))}\int_{x}^{\overline{x}}\left\{\int_{\underline{x}}^{w} G(y)^{n-2}\,dy\right\} g(w)\,dw\right]$$

and $\beta_I^{/}(x) < 0$ (Appendix A.1)

Next we proceed to check whether the derived bid function satisfies the second order condition for maximisation of expected payoff and find that it does (Appendix A.3).

3 Conclusion

We have considered a sequential auction where the bidders have waiting costs. The initial value of the object is common knowledge and the waiting costs are private information to the bidders. We consider a two-stage sequential auction. Our results suggest that if the types are symmetrically, independently and identically distributed among bidders, who are risk neutral, then there exists a symmetric perfect Bayesian equilibrium where the bids are increasing and decreasing functions of the types respectively in the first and the second stage auctions. It is interesting to note that this result somewhat resembles the “weakness leads to aggression” outcome which is observed in case of auctions involving asymmetric types of bidders, although it has been derived for a symmetric, independent private valuations framework. Further scope of research in this direction lies, first, in investigating the bidding behaviour when the types, instead of being private information, are of a common value nature. Second, it would be interesting to study the bidding behaviour when both, the valuation for the object as well the waiting cost, are private information.

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Appendix

A.1

The distribution of the lowest order statistic X_1 for $(n-1)$ values of the random variable X (distributed according to the probability distribution function $G(.)$) is given by

$$G_1(x) = 1 - (1 - G(x))^{n-1}$$

From this we can calculate that the probability for all the $(n-2)$ bidders' types being higher than the lowest type x_1, is

$$1 - G_1(x_1) = (1 - G(x_1))^{n-2}$$

and analogously the probability for all the $(n-1)$bidders' types being less z, is

$$\left(1-G\left(z\right)\right)^{n-1}$$

A.2

We can see that

$$\beta_I^{/}\left(x\right)=\frac{(n-1)g(x)}{(1-G(x))^{n-1}}\left[\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy-\frac{(n-1)}{(1-G(x))}\int_{x}^{\overline{x}}\left\{\int_{\underline{x}}^{w}G\left(y\right)^{n-2}dy\right\}g\left(w\right)dw\right]$$
$$=\frac{(n-1)g(x)}{(1-G(x))^{n}}\left[\left(1-G\left(x\right)\right)\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy-\left(n-1\right)\int_{x}^{\overline{x}}\left\{\int_{\underline{x}}^{w}G\left(y\right)^{n-2}dy\right\}g\left(w\right)dw\right]$$

Now,

$$\begin{aligned}
&\int_{x}^{\overline{x}}\left\{\int_{\underline{x}}^{w}G\left(y\right)^{n-2}dy\right\}g\left(w\right)dw\\
&=\left[\int_{\underline{x}}^{w}G\left(y\right)^{n-2}dy\int g\left(w\right)dw\right]_{x}^{\overline{x}}-\int_{x}^{\overline{x}}\left\{G\left(w\right)^{n-2}\int g\left(w\right)dw\right\}dw\\
&=\left[G\left(w\right)\int_{\underline{x}}^{w}G\left(y\right)^{n-2}dy\right]_{x}^{\overline{x}}-\int_{x}^{\overline{x}}\left\{G\left(w\right)^{n-2}G\left(w\right)\right\}dw\\
&=G\left(\overline{x}\right)\int_{\underline{x}}^{\overline{x}}G\left(y\right)^{n-2}dy-G\left(x\right)\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy-\int_{x}^{\overline{x}}\left\{G\left(y\right)^{n-1}\right\}dy\\
&=\int_{\underline{x}}^{\overline{x}}G\left(y\right)^{n-2}dy-G\left(x\right)\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy-\int_{x}^{\overline{x}}\left\{G\left(y\right)^{n-1}\right\}dy
\end{aligned}$$

We denote, $\Phi\left(x\right)=\left(1-G\left(x\right)\right)\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy-\left(n-1\right)\int_{x}^{\overline{x}}\left\{\int_{\underline{x}}^{w}G\left(y\right)^{n-2}dy\right\}g\left(w\right)dw$

We can rewrite this as

$$\Phi\left(x\right)=\left(1-G\left(x\right)\right)\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy-\left(n-1\right)\left[\int_{\underline{x}}^{\overline{x}}G\left(y\right)^{n-2}dy-G\left(x\right)\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy-\int_{x}^{\overline{x}}\left\{G\left(y\right)^{n-1}\right\}dy\right]$$
$$=\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy-G\left(x\right)\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy-\left(n-1\right)\int_{\underline{x}}^{\overline{x}}G\left(y\right)^{n-2}dy+\left(n-1\right)G\left(x\right)\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy+\left(n-1\right)\int_{x}^{\overline{x}}\left\{G\left(y\right)^{n-1}\right\}dy$$
$$=\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy+\left(n-2\right)G\left(x\right)\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy+\left(n-1\right)\int_{x}^{\overline{x}}\left\{G\left(y\right)^{n-1}\right\}dy-\left(n-1\right)\int_{\underline{x}}^{\overline{x}}G\left(y\right)^{n-2}dy$$

Now, at $x=\underline{x}$, we have $\Phi\left(x\right)<0$ and at $x=\overline{x}$, we have $\Phi\left(x\right)=0$. We can also easily check that,

$$\begin{aligned}
&\Phi^{/}\left(x\right)=G\left(x\right)^{n-2}+\left(n-2\right)G\left(x\right)^{n-1}-\left(n-1\right)G\left(x\right)^{n-1}\\
&=G\left(x\right)^{n-2}-G\left(x\right)^{n-1}+\left(n-2\right)g\left(x\right)\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy\\
&=G\left(x\right)^{n-2}\left(1-G\left(x\right)\right)+\left(n-2\right)g\left(x\right)\int_{\underline{x}}^{x}G\left(y\right)^{n-2}dy>0\,\forall x\in\left[\underline{x},\overline{x}\right]
\end{aligned}$$

Thus combining the conditions that, $\Phi\left(x\right)<0$ at $x=\underline{x}$, $\Phi\left(x\right)=0$ at $x=\overline{x}$ and the fact that $\Phi^{/}\left(x\right)>0\,\forall x\in\left[\underline{x},\overline{x}\right]$we can readily infer that $\beta_I^{/}\left(x\right)<0\,\forall x\in\left[\underline{x},\overline{x}\right)$.

A.3

The expected payoff function, after substituting for the bid function in the first stage, can be written as

$$
\begin{aligned}
\Pi_1(z;x) &= (1-G(z))^{n-1}\left(V - V + \tfrac{n-1}{(1-G(z))^{n-1}} \int_z^{\overline{x}} \left\{ \int_{\underline{x}}^{w} G(y)^{n-2}\, dy \right\} g(w)\, dw \right) \\
&\quad + (n-1)\, G(z) \int_{\underline{x}}^{x} G(y)^{n-2}\, dy \\
&= (n-1) \int_z^{\overline{x}} \left\{ \int_{\underline{x}}^{w} G(y)^{n-2}\, dy \right\} g(w)\, dw + (n-1)\, G(z) \int_{\underline{x}}^{x} G(y)^{n-2}\, dy \\
&= (n-1) \left[\int_z^{\overline{x}} \left\{ \int_{\underline{x}}^{w} G(y)^{n-2}\, dy \right\} g(w)\, dw + G(z) \int_{\underline{x}}^{x} G(y)^{n-2}\, dy \right]
\end{aligned}
$$

Now from the first order condition for profit maximisation, we obtain,

$$
\begin{aligned}
&\tfrac{\partial \Pi_1}{\partial z} = 0 \\
&\Rightarrow (n-1)\left[-\left\{ \int_{\underline{x}}^{z} G(y)^{n-2}\, dy \right\} g(z) + g(z) \int_{\underline{x}}^{x} G(y)^{n-2}\, dy \right] = 0 \\
&\Rightarrow g(z) \left[\int_{\underline{x}}^{x} G(y)^{n-2}\, dy - \int_{\underline{x}}^{z} G(y)^{n-2}\, dy \right] = 0 \\
&\Rightarrow g(z) \int_z^{x} G(y)^{n-2}\, dy = 0 \\
&\Rightarrow z = x
\end{aligned}
\tag{3.1}
$$

From the second order condition, we obtain,

$$
\frac{\partial^2 \Pi_1}{\partial z^2}\Big|_{z=x} = -g(x)\, G(x)^{n-2} < 0
$$

$$
\beta_I - \beta_{II} = c - \tfrac{(n-1)}{(1-G(x))^{n-1}} \int_x^{\overline{x}} \left\{ \int_{\underline{x}}^{w} G(y)^{n-2}\, dy \right\} g(w)\, dw + \tfrac{1}{G(x)^{n-2}} \int\limits_{\underline{x}}^{x} \left[G(y)^{n-2} \right] dy
$$