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Continuous and strategyproof mechanisms

Ranojoy Basu
Assistant Professor
Indian Institute of Management Udaipur
Balicha, Udaipur - 313001, Rajasthan, India
Email: ranojoy.basu@iimu.ac.in

Conan Mukherjee*
Assistant Professor
Economics Group
Indian Institute of Management Calcutta
D. H. Road, Joka, Kolkata 700 104, West Bengal, India
Email: conan.mukherjee@iimcal.ac.in
(* Corresponding Author)

Indian Institute of Management Calcutta
Joka, D.H. Road
Kolkata 700104

URL: http://facultylive.iimcal.ac.in/workingpapers

Continuous and strategyproof mechanisms *

Ranojoy Basu[†1] and Conan Mukherjee[‡2]

[1]Economics and Finance area, Indian Institute of Management Udaipur

[2]Economics group, Indian Institute of Management Calcutta

Abstract

We introduce a novel notion of continuity of mechanisms, and present a complete characterization result which shows that: the class of VCG (Vickrey [26], Clarke [3], Groves [6]) mechanisms is the only class of strategyproof mechanisms that satisfy (weak) agent sovereignty, non-bossiness in decision, and continuity. We find that efficient mechanisms are actually a well-behaved subset of continuous strategyproof mechanisms.

JEL classification: C72; C78; D71; D63

Keywords: strategyproofness, continuity, VCG mechanism

1 Introduction

The seminal works of Green and Laffont [5], and Holmström [9] establish VCG (Vickrey [26], Clarke [3], Groves [6]) mechanisms as the only strategyproof mechanisms that ensure decision efficient (aggregate social welfare maximizing) outcome in a large class of mechanism settings. In this paper, we present a new characterization of VCG mechanisms for the single object allocation problem *without* applying the axiom of decision efficiency, in a private information quasi-linear preference setting. This model can be applied to a variety of social decisions involving assignment of a license, a house, a plot of land or

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†ranojoy.basu@iimu.ac.in

‡conanmukherjee@gmail.com

airport landing rights etc. The standard bilateral trade mechanism design problem, too, is a two agent special case of this model.

The novelty of our results follows mainly from a new concept of '*continuity*' of mechanisms. In particular, we define a mechanism to be continuous if for any sequence of profiles where the object allocation decision remain unchanged, either the same decision holds in the limit or the agents are indifferent between winning and losing the object. Note that the desirability of continuity of mechanisms has been around for a long time in the literature. Thomson [25] notes that the continuity property has appealing strategic as well as ethical characteristics. In highlighting its strategic value, he states that "*...a discontinuous rule is likely to be manipulable in undetectable ways*"; while in specifying its ethical value, he contends that continuity rules out unfair situations where small changes in underlying preferences (which may arise due to involuntary inaccurate reporting) result in stark changes in agents' welfare.

This paper completely characterizes the class of VCG mechanisms using strategyproofness, continuity and the following restrictions:[1]

- *weak agent sovereignty* which requires that an agent be always able to get the object by reporting a suitable valuation, and
- *non-bossiness in decision* which requires that no agent influence other's assignment decision (excluding transfers) without affecting her own.

Note that agent sovereignty is a desirable ethical property, which in words of Moulin [14], "*is reminiscent of the citizen sovereignty of classical social choice.*" In the same vein, as argued in Thomson [25], non-bossiness of decision may be interpreted in strategic terms as it discourages collusive practices where agents may form groups, and misreport in a manner that changes the allotment decision to benefit any one member of the group while not making any other member worse off. Remarkably, we find that any continuous strategyproof mechanisms satisfying the aforementioned properties must be decision efficient.

[1] We also use a mild boundary condition requiring that agents bidding zero not get an object when some other agent has bid a positive value.

The paper is organized as follows. Section 2 presents the relation to literature, while Section 3 presents the model and axioms. Sections 4 and 6 present the main results and the conclusion, respectively. Section 5 presents the arguments for independence of the axioms uses, and Section 7 presents the appendix containing auxiliary results.

2 Relation to literature

The problem of indivisible object allocation with monetary transfers is a well-studied one. However, most papers in literature attempt to characterize a subset of strategyproof mechanisms that adhere to acceptable notions of fairness. For example, Tadenuma and Thomson [24] study the single object allocation problem and show that no proper sub-solution of the no-envy solution satisfies consistency, while Ohseto [20] considers a multiple identical objects allocation problem with unit demand and characterizes the class of non-envious and strategyproof mechanisms. Svensson and Larsson [23] study a heterogeneous object allocation problem with unit demand and show that any strategyproof, non-bossy and neutral allocation rule must be serially dictatorial. Pápai [21] and Yengin [27] study a more general heterogeneous object allocation problem and characterize the classes of non-envious VCG mechanisms and egalitarian-equivalent VCG mechanisms, respectively. Ashlagi and Serizawa [1] characterize a subset of VCG mechanisms as the unique class of strategyproof mechanisms that satisfy anonymity in welfare and individual rationality in a *multiple* identical object allocation problem. Mukherjee [16] builds upon their result and characterizes the subset of VCG mechanisms which is the unique class of strategyproof mechanisms satisfying anonymity in welfare (or no-envy) in the same setting. Hashimoto and Saitoh [8] establish a similar result in the context of queueing games.

The paper closest to ours is Ashlagi and Serizawa [1]. It considers a multiple identical object allocation problem with money where *no positive* transfers can be made, and shows, that the only strategyproof mechanism that satisfies anonymity in welfare and individual rationality is the Vickrey mechanism.[2] Note that, unlike this paper, the very setting of

[2] Vickrey mechanism is a uniform price auction where the highest bidders win objects and pay the

Ashlagi and Serizawa [1], rules out positive transfers, and so, excludes a subset of VCG mechanisms. Further, Ashlagi and Serizawa [1] presents a characterization of decision efficiency that is independent of the one presented in this paper. They show that every strategyproof mechanism satisfying anonymity in welfare and individual rationality must be decision efficient when no positive transfers can ever be made.[3] However, in their paper, the imposition of anonymity in welfare implied that (i) the transfer functions would be independent of agent identities, and (ii) the transfer functions of each agent would be symmetric. In contrast, the present paper uses the novel continuity condition, which ensures that any well behaved strategyproof mechanism satisfying the aforementioned properties - must be decision efficient (even when positive transfers are allowed). This result allows us to completely characterize the *full* class of VCG mechanisms (including those not satisfying (i) and (ii)) *without* using decision efficiency.

To the best of our knowledge, there is no other paper that characterizes the complete class of VCG mechanisms without use of the restriction of decision efficiency.

3 Model

Consider an assignment problem where a single indivisible object must be allotted to any one member from the agent set $N = \{1, \ldots, n\}, n \geq 3$ using monetary transfers. Each agent i has a private valuation $v_i \geq 0$ for the object. A mechanism μ is a tuple (d, τ) such that at any reported profile of valuations $v \in \mathbb{R}_+^N$, each agent i is allocated a transfer $\tau_i(v) \in \mathbb{R}$ and a decision $d_i(v) \in \{0, 1\}$ where $\sum_{i \in N} d_i(v) = 1$. The notation $d_i(v) = 1$ denotes agent i getting the object, while $d_i(v) = 0$ stands for i not getting the object. [4] Define $w(v)$ to be the agent getting the object at any profile v.[5] The utility to agent i with a true valuation of v_i at any reported profile $v' \in \mathbb{R}_+^N$ from a mechanism μ is given by $u(d_i(v'), \tau_i(v'); v_i) = v_i d_i(v') + \tau_i(v')$. Let $\forall\, i \in N$, $\forall\, S \subseteq N$ with $|S| > 1$, $\forall\, v \in \mathbb{R}_+^N$,

greatest losing bids as price.

[3] Mukherjee [16] strengthens their result to show that any strategyproof mechanism satisfying anonymity in welfare must be decision efficient *without* any restriction on transfers.

[4] Note that we assume that the object is allocated at each profile of reported valuations. This premise has also been used by Athey and Miller [2], Miller [11], Hagerty and Rogerson [7], Drexler and Kleiner [4] and Shao and Zhou [22].

[5] We often refer to this agent $w(v)$ as the winner at profile v in the text.

$v_{-i} := (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$, $v_{-S} := (v_i)_{i \in N \setminus S}$ and $v_S := (v_i)_{i \in S}$. Also, define for all $x \geq 0$, and all $t \in \{1, \ldots, n\}$, $\bar{x}^t := (x, x, x, \ldots, x) \in \mathbb{R}^t_+$. Finally, for any $\delta > 0$, any $t \in \mathbb{N}$, and any $y \in \mathbb{R}^t_+$; let $\mathcal{N}_\delta(y) := \{z \in \mathbb{R}^t_+ | \delta > ||y - z||\}$ where $||.||$ denotes the Euclidean norm.

We begin by defining the class of VCG mechanisms in the current setting.

Definition 1 *A mechanism $\mu^V = (d^V, \tau^V)$ is a VCG mechanism if and only if for all $i \in N$, and all $v \in \mathbb{R}^N_+$,*

- $d^V_i(v) = 1 \Longrightarrow v_i \geq v_j, \forall\, j \neq i.$
- *There exists a function $h_i : \mathbb{R}^{N \setminus \{i\}} \mapsto \mathbb{R}$ such that*

$$\tau^V_i(v) = \begin{cases} -\max_{j \neq i} v_j + h_i(v_{-i}) & \text{if } d_i(v) = 1 \\ h_i(v_{-i}) & \text{otherwise} \end{cases}$$

Let $\mathcal{M}^{VCG}$ denote the class of all VCG mechanisms in this setting.

Now, we define a class of mechanisms that are well behaved in the following sense.

Definition 2 *Let Γ be the set of mechanisms $\mu = (d, \tau)$ that satisfy the following properties:*

(a) Continuity *For any $\zeta \in \{0, 1\}$, any $i \in N$ and any sequence of profiles $\{v^k\}$ that converges to $\tilde{v} \in \mathbb{R}^N_+$, whenever $d_i(v^k) = \zeta$ for all k,*

$$d_i(\tilde{v}) \neq \zeta \Longrightarrow u(1, \tau_i(\tilde{v}); \tilde{v}_i) = u(0, \tau_i(\tilde{v}); \tilde{v}_i).$$

(b) Reasonability *For any $i \in N$ and any $v \in \mathbb{R}^N_+$, if there exists another agent $j \neq i$ with $v_j > 0$, then*

$$d_i(0, v_{-i}) = 0.$$

(c) Weak agent sovereignty *For any $i \in N$ and any $v \in \mathbb{R}^N_+$, there exists an $x^{v_{-i}} \geq 0$ such that*

$$d_i(x^{v_{-i}}, v_{-i}) = 1.$$

(d) Non-bossiness in decision *For any* $i \in N$*, any* $v \in \mathbb{R}^N_+$*, and any* $x \geq 0$*,*

$$d_i(v) = d_i(x, v_{-i}) \Longrightarrow \forall\, j \neq i, d_j(v) = d_j(x, v_{-i}).$$

The condition **(a)** above is a continuity condition. It requires that for all convergent sequences of profiles, if (i) some agent i gets the object at all member profiles of the sequence, and (ii) some other agent $j \neq i$ gets the object at the limit profile; then the transfers of i and j at the limit profile should make them indifferent between winning and losing the object.[6] The condition **(b)** is a boundary condition that rules out any agent getting the object by reporting a zero valuation. Note that this idea, in itself, represents a desirable property which requires no agent should get the object when she reports no desire for it. However, we impose a weaker restriction, requiring that no agent get the object by reporting zero valuation *only when there is another agent who reports a positive valuation.*

The condition **(c)** of 'weak agent sovereignty' presents the idea that each agent must *always* be able to impact the allotment process in her favour, by reporting a suitable value, should she find it preferable to do so. This restriction also been used in other mechanism design settings by Lavi, Mualem and Nisan [10] (who refer to this restriction as '*player decisiveness*'), Moulin and Shenker [15] and Marchant and Mishra [12].[7] Finally, the condition **(d)** presents a version of non-bossiness which requires that no agent be able to change any other agent's allotment decision, without changing her own decision. As argued by Thomson [25], non-bossiness of decision, in company of strategyproofness, embodies strategic restraints to collusive practices where agents form groups to misreport in a manner that changes the allotment decision to benefit one member of the group while not making any other member worse off. This condition has been used in other mechanism design settings by Nath and Sen [18] and Mishra and Quadir [13].

In this paper, we look for mechanisms in Γ that are immune to strategic manipulation

[6] The same implication must hold for any agent who does not get the object at any member profile of the convergent sequence of profiles, but gets the object at the limit profile.

[7] Note that the restriction **(b)** would no longer be needed for our results if we use a stronger version of agent sovereignty that requires that for all i and all $v_{-i} \in \mathbb{R}^{N\setminus\{i\}}$, there exist $x^{v_{-i}}, y^{v_{-i}} \geq 0$ such that $d_i(x^{v_{-i}}, v_{-i}) \neq d_i(y^{v_{-i}}, v_{-i})$.

in reporting. In particular, we use the popular strategic axiom of strategyproofness, which eliminates any incentive to misreport on an individual level. It is defined as follows.

Definition 3 A mechanism $\mu = (d, \tau)$ satisfies *strategyproofness* (SP) if $\forall i \in N$, $\forall v, v' \in \mathbb{R}^N_+$ such that $v_{-i} = v'_{-i}$,

$$u(d_i(v), \tau_i(v); v_i) \geq u(d_i(v'), \tau_i(v'); v_i)$$

Thus, a strategyproof mechanism guarantees that revealing the true valuation is a weakly dominant strategy for each agent in the simultaneous move game that ensues from the mechanism. The purpose of this paper is to show that the class mechanisms in Γ that satisfy SP is same as $\mathcal{M}^{VCG}$.

4 Results

We start by stating a well-known characterization of strategyproof mechanisms.

Result 1 A mechanism $\mu = (d, \tau)$ satisfies SP if and only if $\forall\, i \in N$ and $\forall\, v \in \mathbb{R}^N_+$, there exist real valued functions $K^\mu_i : \mathbb{R}^{N\setminus\{i\}}_+ \mapsto \mathbb{R}$ and $T^\mu_i : \mathbb{R}^{N\setminus\{i\}}_+ \mapsto \mathbb{R} \cup \{\infty\}$ such that

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > T^\mu_i(v_{-i}) \\ 0 & \text{if } v_i < T^\mu_i(v_{-i}) \end{cases} \quad \text{and} \quad \tau_i(v) = \begin{cases} K^\mu_i(v_{-i}) - T^\mu_i(v_{-i}) & \text{if } d_i(v) = 1 \\ K^\mu_i(v_{-i}) & \text{if } d_i(v) = 0 \end{cases}$$

Proof: The result follows from Proposition 9.27 in Nisan [19] and Lemma 1 in Mukherjee [17]. □

Note that Result 1 allows for arbitrary tie-breaking in allocation decision of the object at any profile $v \in \mathbb{R}^N_+$ such that there exists an agent $i \in N$ with $v_i = T^\mu_i(v_j)$. In this paper, *without loss of generality*, we assume a tie-breaking rule in favour of agent 1 such that: for any profile $v \in \mathbb{R}^N_+$,

$$v_1 = T^\mu_1(v_{-1}) \Longrightarrow d_1(v) = 1.$$

Thus, for any agent $i \neq 1$, this tie breaking rule does not allocate the object to i at any valuation profile where her reported value equals her threshold. Note that, our assumption of object being allocated at all profiles, in conjunction with any tie-breaking rule, requires threshold functions to be sufficiently well behaved so that whenever $v_i = T_i^{\mu}(v_{-i}), i \neq 1$, there exists a $j \neq i$ such that $v_j > T_j^{\mu}(v_{-j})$.

We begin by presenting the following theorem which plays an important role in establishing our proof. It states that for any strategyproof mechanism in Γ, the threshold functions $\{T_i^{\mu}(.)\}_{i \in N}$ of Result 1 must be continuous, and have finite non-negative image at all points in $\mathbb{R}_+^{n-1}$.

Theorem 1 *For any mechanism $\mu \in \Gamma$ that satisfies SP,*

1. $T_i^{\mu}(z) \in [0, \infty)$ *for all* $z \in \mathbb{R}_+^{N \setminus \{i\}}$, *and all* $i \in N$.
2. $\lim_{v_{-i} \to z} T^{\mu}(v_{-i}) = T_i^{\mu}(z)$ *for all* $z \in \mathbb{R}_+^{N \setminus \{i\}}$, *and all* $i \in N$.

Proof: Fix any mechanism $\mu = (d, \tau) \in \Gamma$. Fix any i and any $v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$ such that $v_{-i} \neq \bar{0}^{n-1}$. If $T_i^{\mu}(v_{-i}) < 0$ then, by Result 1, $d_i(0, v_{-i}) = 1$ which contradicts condition **(b)** of Definition 2. Also, if $T_i^{\mu}(v_{-i}) = \infty$, then $d_i(x, v_{-i}) = 0$ for all $x \geq 0$ which contradicts condition **(c)** of definition 2. Now consider the point $\bar{0}^{n-1}$. Note that arguing as above we can show that condition **(c)** implies $T_i^{\mu}(\bar{0}^{n-1}) < \infty$. Consider the possibility that $T_i^{\mu}(\bar{0}^{n-1}) < 0$ which implies that $d_i(\bar{0}^n) = 1$. Now, by condition **(b)**, $d_i(0, v_{-i}) = 0$ whenever $v_{-i} >> \bar{0}^{n-1}$. And so, for any sequence of profiles $\{v^k\}$ that converges to $\bar{0}^n$, such that for all k, $v_i^k = 0$ and $v_{-i}^k >> \bar{0}^{n-1}$; we have $d_i(v^k) = 0$ but $d_i(\bar{0}^n) = 1$, and so, condition **(a)** of Definition 2 implies that $0 - T_i^{\mu}(\bar{0}^{n-1}) = 0$, which is a contradiction to our supposition. Hence, $T_i^{\mu}(\bar{0}^{n-1}) \geq 0$, and so, the condition (1) follows.

To prove result (2), fix any agent i, and any sequence $\{z^k\}$ such that $z^k \in \mathbb{R}^{N \setminus \{i\}}$ for all k, $\{z^k\} \to z^*$. Suppose that the sequence $\{T_i^{\mu}(z^k)\}$ does not converge to $T_i^{\mu}(z^*)$. By the aforementioned result (1), $T_i^{\mu}(z^*) \in [0, \infty)$. Suppose that the sequence $\{T_i^{\mu}(z^k)\}$ is unbounded above. Hence, there exists a monotone increasing subsequence $\{T_i^{\mu}(z_l^k)\}_{l=1}^{\infty}$ such that it is *properly divergent*, that is, with some abuse of notation, $\{T_i^{\mu}(z_l^k)\} \to \infty$. Therefore, there exists an $M^* \in \mathbb{N}$ such that for all $l > M^*$, $T_i^{\mu}(z_l^k) > T_i^{\mu}(z^*) + 1$. Hence, for any sequence of profiles $\{v^t\}$ where for all t, $v_i^t = T_i^{\mu}(z^*) + 1$ and $v_{-i}^t = z_{M*+t}^k$, by Result

1, $d_i(v^t) = 0$. Further, by construction, $\{v^t\}$ converges to v^* where $v_i^* = T_i^\mu(z^*) + 1$, and $v_{-i}^* = z^*$; and so, by Result 1, $d_i(v^*) = 1$. Thus, by **(a)** of definition 2, $u(1, \tau_i(v^*); v_i^*) = u(0, \tau_i(v^*); v_i^*)$ implying that $T_i^\mu(z^*) + 1 = T_i^\mu(z^*)$, which is a contradiction.

Therefore, by result (1), we can infer that $\{T_i^\mu(z^k)\}$ is a bounded sequence, and so, it must have a convergent subsequence. *To simplify notation*, we assume without loss of generality that $\{T_i^\mu(z^{k^l})\}$ is a convergent sequence that converges to some $\beta \geq 0$. By supposition, $\beta \neq T_i^\mu(z^*)$. Suppose, without loss of generality, $\beta > T_i^\mu(z^*)$ and fix any $x \in (T_i^\mu(z^*), \beta)$. Note that there exists a (tail) subsequence $\{T_i^\mu(z^{k^l})\} \subseteq (\beta - \epsilon, \beta + \epsilon)$ for some particular $\epsilon \in (0, \beta - x)$, such that $\{T_i^\mu(z^{k^l})\} \to \beta$. Therefore, we can construct a sequence of profiles $\{v^l\}$ such that for all l, $v_i^l = x$ and $v_{-i}^l = z^{k^l}$. Now, since $\{z^k\}$ converges to z^*, the subsequence $\{z^{k^l}\}$ must also converge to z^*, and so, $\{v^l\}$ converges to $\bar{v}$ where $\bar{v}_i = x$ and $\bar{v}_{-i} = z^*$. Therefore, for all l, $v_i^l = x < T_i^\mu(v_{-i}^l)$ which implies that $d_i(v^l) = 0$; but in limit $x > T_i^\mu(z^*)$ which implies that $d_i(\bar{v}) = 1$. Again, by **(a)**, $u(1, \tau_i(\bar{v}); \bar{v}_i) = u(0, \tau_i(\bar{v}); \bar{v}_i)$ implying that $x = T_i^\mu(z^*)$, which is a contradiction to our construction of x. Hence the condition (2) follows. □

In the following theorem, we show that any strategyproof mechanism in Γ must be efficient. But first, we define the notion of efficient mechanism in our model.

Definition 4 *A mechanism $\mu = (d, \tau)$ satisfies efficiency (EFF) if for all $v \in \mathbb{R}_+^N$, $\sum_{i \in N} d_i(v) v_i$ solves the problem*

$$\max \left\{ \sum_{i \in N} \hat{d}_i(v) v_i : \hat{d} \in \{0, 1\}^n \right\}.$$

It is easy to see that any efficient mechanism in our model is efficient if and only if, it allots the object to an agent reporting the highest valuation, at all profiles. As mentioned earlier, the following theorem specifies a connection between strategyproofness and efficiency in our model.

Theorem 2 *If a mechanism $\mu = (d, \tau) \in \Gamma$ satisfies SP then it satisfies EFF.*

Proof: See Appendix. □

Now, we present the main result of this paper which states that the only strategyproof mechanisms in Γ are the VCG mechanisms.

Theorem 3 *$\mathcal{M}^{VCG}$ is the unique class of mechanisms in Γ that satisfy SP.*

Proof: To prove this result, we need to show that any mechanism $\mu \in \Gamma$ satisfies SP if and only if $\mu = \mu^V$. The proof of necessity follows from Result 1 and Theorem 2 above. To see the sufficiency, we simply need to show that $\mu^V \in \Gamma$.[8] It is easy to see that for μ^V: (i) the object is given, at all reported profiles, to any one of the highest bidders implying that μ^V satisfies conditions **(b)** & **(d)**; and (ii) every agent can report a value greater that all her competitors' reported values to get the object, implying than μ^V satisfies condition **(c)**. To see that μ^V also satisfies the continuity condition **(a)**, consider any convergent sequence of profiles $\{v^t\}$ with limit at $\bar{v} \geq 0$, such that $d^V(v^t)$, and hence, $w(v^t)$ remains unchanged with t. Hence, we can define an agent $\bar{w} \in N$ such that $w(v^t) = \bar{w}$ for all $t \in \mathbb{N}$. Therefore, by definition of μ^V, for all t, $v^t_{\bar{w}} \geq v^t_j$ for all $j \neq \bar{w}$, and so, in limit $\bar{v}_{\bar{w}} \geq \bar{v}_j$ for all $j \neq \bar{w}$. Now if $\bar{w} \neq w(\bar{v})$, then by definition of μ^V, for all $t \in \mathbb{N}$: (i) $d^V_i(v^t) = d^V_i(\bar{v})$ for all $i \in N \setminus \{\bar{w}, w(\bar{v})\}$, (ii) $d^V_i(v^t) \neq d^V_i(\bar{v})$ for all $i \in \{\bar{w}, w(\bar{v})\}$, and (iii) $\bar{v}_{\bar{w}} = \bar{v}_{w(\bar{v})}$ implying that $\tau^V_{w(\bar{v})} = -\bar{v}_{\bar{w}} + h_{w(\bar{v})}(v_{-w(\bar{v})})$. The third statement implies that for both $i \in \{\bar{w}, w(\bar{v})\}$, $u(1, \tau^V_i(\bar{v}); \bar{v}_i) = u(0, \tau^V_i(\bar{v}); \bar{v}_i) = h_i(v_{-i})$, and so, we get that μ^V satisfies condition **(a)**. Hence, $\mu^V \in \Gamma$. □

5 Independence of axioms

In this section, we show independence of the axioms used in our model. We do so by providing an example of a mechanism that violates each one of the axioms while satisfying all others.

Note that there are five axioms that we use in deriving our results. They are: continuity (C), reasonability (R), non-bossiness in decision (NBD), strategyproofness (SP), weak agent sovereignty (WAS). We establish the independence of these axioms providing

[8]It is well known that μ^V is strategyproof.

five examples of mechanisms which fail to satisfy one of these axioms, while satisfying all other properties. These examples are as follows:

$\neg$ **NBD** Consider a setting where $N = \{1, 2, 3\}$, and a mechanism of the kind described in Result 1 such that: (i) for any $v \in \mathbb{R}^N_+$,

$$T_1(v_2, v_3) = v_2 + v_3, T_2(v_1, v_3) = \max\{v_1 - v_3, v_3\}, T_3(v_1, v_2) = \max\{v_1 - v_2, v_2\},$$

(ii) ties are broken in favour of agent 3, and (iii) losers receive zero transfers. Note that $d(9, 5, 3) = (1, 0, 0)$, but $d(9, 5, 4.5) = (0, 1, 0)$; implying that this mechanism does not satisfy NBD.

It is easy to see that this mechanism satisfies C, R, SP, and WAS. Further, note that whenever $v_1 < T_1(v_{-1})$, the higher bidder $i \in \{1, 2\}$ gets the object, and so, this mechanism allocates the object at all profiles. Finally, consider sequence of profiles $(v^k) \to \bar{v}$ such that for all k, (without loss of generality) $d_2(v^k) = 1$. Further, suppose that $d_2(\bar{v}) = 0$. Then, by construction, $v_2^k \geq T_2(v_{-2}^k)$ for all k, which implies that $\bar{v}_2 = \max\{\bar{v}_1 - \bar{v}_3, \bar{v}_3\}$, further implying that $u(1, \tau_2(\bar{v}); \bar{v}_2) = u(0, \tau_2(\bar{v}); \bar{v}_2)$.

$\neg$ **C** Consider a setting where $N = \{1, 2\}$, and a mechanism of the kind described in Result 1 such that for all $v \in \mathbb{R}^N_+$:

$$T_1(v_2) = \begin{cases} 2v_2 & v_2 \geq 50 \\ v_2 & \text{otherwise} \end{cases} \quad \text{and } T_2(v_1) = \begin{cases} v_1 & v_1 < 50 \\ 50 & v_1 \in [50, 100) \\ \frac{v_1}{2} & v_1 \geq 100 \end{cases}$$

and ties are broken in favour of agent 1 with losers receiving zero transfers. It is easy to see by Theorem 1 that this mechanism violates C. It is also easy to check that the object is allocated at all profiles, and hence, this mechanism satisfies NBD. Finally, one can easily check that this mechanism satisfies R, SP and WAS.

$\neg$ **R** Consider a setting where $N = \{1, 2\}$, and a mechanism of the kind described in

Result 1 such that for all $v \in \mathbb{R}^N_+$:

$$T_1(v_2) = v_2 - 5 \text{ and } T_1(v_2) = v_1 + 5$$

with ties broken arbitrarily and losers receiving zero transfers. It is easy to see that agent 1 is allocated an object at the profile $(0, 1)$, which violates R.

Further, it is easy to see that this mechanism allocates the object at all profiles, and hence, satisfies NBD, SP and WAS. Finally, to see that this mechanism satisfies C, consider sequence of profiles $(v^k) \to \bar{v}$ such that for all k, (without loss of generality) $d_2(v^k) = 1$. If $d_2(\bar{v}) = 0$ then, by construction, $v_2^k \geq T_2(v_{-2}^k)$ for all k, which implies that $\bar{v}_2 = \bar{v}_1 + 5$, further implying that $u(1, \tau_2(\bar{v}); \bar{v}_2) = u(0, \tau_2(\bar{v}); \bar{v}_2)$.

$\neg$ **WAS** Consider a setting where $N = \{1, 2\}$, and a mechanism of the kind described in Result 1 such that:

1. for all $v \in \mathbb{R}^N_{++}$:

$$T_1(v_2) = \max\left\{\frac{5v_2 - 1}{v_2}, 0\right\} \text{ and } T_2(v_1) = \frac{1}{\max\{0, 5 - v_1\}},$$

2. for any $\nu > 0$; $d(0, \nu) = (0, 1)$, $d(\nu, 0) = (1, 0)$.
3. ties are broken arbitrarily, and losers receive zero transfers.

Note that agent 2 can never get the object agent 1 reports a valuation 5, implying that WAS is violated. Further, the object is allocated at all profiles, and hence, the mechanism satisfies SP as well as NBD. Also, it trivially, follows from the construction that the mechanism satisfies R.

Now, to check for C, consider the for sequence of profiles (v^k) converging to a limit $\bar{v}$ such that $d_1(v^k) = 1$ for all k, and $d_1(\bar{v}) = 0$. Now, there are two possibilities: (i) $\bar{v} \in \mathbb{R}^N_{++}$, and (ii) $\bar{v} \notin \mathbb{R}^N_{++}$. In case of (i), there must exist a subsequence $(v^{k^l}) \to \bar{v}$, such that for all l, $v^{k^l} \in \mathbb{R}^N_{++}$. Hence, by construction, $v_1^{k^l} \geq \max\left\{0, 5 - \frac{1}{v_2^{k^l}}\right\}$ for all l, implying that $\bar{v}_1 = \max\left\{0, 5 - \frac{1}{v_2^{k^l}}\right\} \implies u(1, \tau(\bar{v}); \bar{v}_1) = u(0, \tau(\bar{v}); \bar{v}_1)$. In case of possibility (ii), the same argument follows if $\bar{v}_1 > 0$. If instead, $\bar{v}_1 = 0$, then this equality is trivially satisfied as at the limit profile, agent 1 is indifferent between getting the object and not getting the object.

Now, consider another sequence of profiles (w^k) converging to a limit $\bar{w}$ such that $d_1(w^k) = 0$ for all k, and $d_1(\bar{w}) = 1$. Now, if $\bar{w} \neq (0,0)$, then by construction, $\bar{w}_1 > 0$, and so, we can claim existence of a subsequence $(w^{k^l}) \to \bar{w}$ such that for all l, $w^{k^l} > 0$. Then, arguing as earlier, it follows that $u(1, \tau(\bar{w}); \bar{w}_1) = u(0, \tau(\bar{w}); \bar{w}_1)$. Finally, as argued earlier, if $\bar{w} = (0,0)$, then the same equality follows trivially.

Therefore, arguing in the same manner for agent 2, we get that this mechanism satisfies C.

$\neg$ **SP** Consider the following mechanism in the setting where $N = \{1, 2\}$:

- $\forall\, v \in \mathbb{R}^N_+,\ d_i(v) = \begin{cases} 1 & \text{if } 0 < v_i < v_j \\ 0 & \text{if } 0 < v_j < v_i \end{cases}$.
- for any $i \neq j$, if $v_i = 0, v_j > 0$ then $d_i(v) = 0, d_j(v) = 1$.
- winner at any profile pays the other bid as price for the object, while the losers receive zero transfers.
- ties are broken arbitrarily.

It is easy to see that this mechanism falls outside the class of mechanisms characterized by Result 1, and hence, violates SP. Further, as shown earlier in the previous counterexample, this mechanism satisfies C. Also, it is easy to see that this mechanism allocates the object at all profiles, and hence, satisfies NBD. Finally, this mechanism can be easily seen to satisfy R and WAS.

6 Conclusion

We present a new concept of continuity of mechanisms, and use it to completely characterize the full class of VCG mechanisms, without employing a decision efficiency axiom. We show that VCG mechanisms are the only continuous strategyproof mechanisms that satisfy non-bossiness in decision and agent sovereignty. These results provide new connections between continuity, strategyproofness, and efficiency in a standard mechanism design setting.

It would be difficult, but interesting, to investigate whether the presented results continue to hold for multiple identical indivisible objects, or heterogeneous indivisible objects. We leave these questions for future research.

7 Appendix

7.1 Proof of Theorem 2

The proof relies on the following four lemmata.

Lemma 1 *For any mechanism $\mu = (d, \tau) \in \Gamma$ that satisfies SP,*

1. *For all $v \in \mathbb{R}^N_+$ and any $i \in N$, $v_i > T^\mu_i(v_{-i}) \Longrightarrow \{v_j < T^\mu_j(v_{-j}), \forall\, j \neq i\}$.*

2. *For all $v \in \mathbb{R}^N_+$ and any $i \in N$,*

$$v_i = T^\mu_i(v_{-i}) \Longrightarrow \left\{\exists\, j \neq i \text{ such that } v_j = T^\mu_j(v_{-j}) \text{ and } v_k \leq T^\mu_k(v_{-k}), \forall\, k \neq i, j\right\}.$$

Proof: Fix any mechanism $\mu = (d, \tau) \in \Gamma$ that satisfies SP, and any $v \in \mathbb{R}^N_+$. If there exists $i \neq j \in N$ such that $v_i > T^\mu_i(v_{-i})$ and $v_j = T^\mu_j(v_{-j})$, then by Result 1, for all $k \in N \setminus \{i, j\}$, $v_k \leq T^\mu_k(v_{-k})$. Suppose, without loss of generality, that for all $k \neq i, j$, $v_k < T^\mu_k(v_{-k})$.[9] Now, by continuity of the threshold functions (established by Theorem 1), for any $\epsilon \in (0, v_i - T^\mu_i(v_{-i}))$, there exists $\delta^\epsilon_i > 0$ such that for all $z \in \mathbb{R}^{N\setminus\{i\}}_+$ with $||v_{-i} - z|| < \delta^\epsilon_i$, $T^\mu_i(z) < T^\mu_i(v_{-i}) + \epsilon < v_i$. Similarly, for all $k \neq i, j$, there exists $\delta_k > 0$ such that for all $z \in \mathbb{R}^{N\setminus\{k\}}_+$ with $||v_{-k} - z|| < \delta_k$, $v_k < T^\mu_k(v_{-k}) - \delta_k < T^\mu_k(z)$. Hence, defining $\bar{\delta} := \min\{\delta^\epsilon_i, \{\delta_k\}_{k\neq i,j}\}$ (it is well defined as the number of agents is

[9] The same arguments that follow would work if there is any other agent $l \neq i, j$ such that $v_l = T_l(v_{-l})$. The only difference that would arise would be that $\tilde{\delta}$ would now be defined over all agents $k \neq i, j, l$.

finite), we can infer that there exists a $\nu \in (0, \bar{\delta})$ such that $v_i > T_i^{\mu}(v_j + \nu, v_{-\{i,j\}})$, and $v_k < T_k^{\mu}(v_j + \nu, v_{-\{j,k\}})$, $\forall\, k \neq i, j$. Now, since $\nu > 0$, by Result 1, $d_i(v_j + \nu, v_{-j}) = d_j(v_j + \nu, v_{-j}) = 1$ implying a contradiction to single indivisible object setting. Thus, condition (1) follows.

To establish condition (2) consider the possibility that there exists an $i \in N$ and $v \in \mathbb{R}_+^N$ such that $v_i = T_i^{\mu}(v_{-i})$, and $v_j < T_j^{\mu}(v_{-j})$ for all $j \neq i$. Arguing as above, there exists an $\eta > 0$ such that $v_j < T_j^{\mu}(v_i - \eta, v_{-\{i,j\}})$ for all $j \neq i$. By Result 1, it implies that $d_t(v_i - \eta, v_{-i}) = 0$ for all $t \in N$, which contradicts our supposition that the object must be allocated at all reported profiles. Hence, the condition (2) follows. □

Lemma 2 *If a mechanism $\mu = (d, \tau) \in \Gamma$ satisfies SP, then:*

1. *for any $v \in \mathbb{R}_+^N$ and any $i \in N$, $T_i^{\mu}(v_{-i})$ is non-decreasing for any change in direction of each unit vector.*[10]

2. *for any $x \geq 0$, any $i \in N$, and any $v \in \mathbb{R}_+^n$ such that $v_{-i} = \bar{x}^{n-1}$ and $v_i = T_i^{\mu}(\bar{x}^{n-1})$,*

$$v_j = T_j^{\mu}(v_{-j}), \forall\, j \neq i.$$

Proof: Fix any mechanism $\mu = (d, \tau) \in \Gamma$ that satisfies SP, any $i \neq j \in N$ and any $v_{-\{i,j\}} \in \mathbb{R}_+^{N \setminus \{i,j\}}$.[11] Say there exists $0 \leq v_j^1 < v_j^2$ such that $T_i^{\mu}(v_j^1, v_{-\{i,j\}}) > T_i^{\mu}(v_j^2, v_{-\{i,j\}})$. Fix a $\beta \in (T_i^{\mu}(v_j^2, v_{-\{i,j\}}), T_i^{\mu}(v_j^1, v_{-\{i,j\}}))$, and consider two profiles $v, v' \in \mathbb{R}_+^N$ such that $v_i = v_i' = \beta$, $v_{-i} = (v_j^2, v_{-\{i,j\}})$, and $v_{-i}' = (v_j^1, v_{-\{i,j\}})$. By Result 1, $d_i(v) = 1 \implies d_j(v) = 0$, and so, $d_j(v') = 0$ as $v_j^1 < v_j^2$. But, by construction, $d_i(v') = 0$ which implies a contradiction to **(d)**. Hence, the condition (1) follows.

To establish condition (2), fix any $x \geq 0$, any profile v and any agent i such that $v_i = T_i^{\mu}(\bar{x}^{n-1})$ and $v_{-i} = \bar{x}^{n-1}$. By Lemma 1, there exists an agent $k \neq i$ such that $x =$

[10] Unit vectors are the vectors $e^1, \ldots, e^{n-1} \in \mathbb{R}_+^{n-1}$ such that each $t = 1, \ldots, n-1$ $e_l^t = \begin{cases} 1 & \text{if } t = l \\ 0 & \text{otherwise} \end{cases}$

[11] If $|N| = 2$, then the result would follow trivially from Lemma 1.

$v_k = T_k^\mu(T_i^\mu(v_{-i}), v_{-\{i,k\}}) = T_k^\mu(T_i^\mu(\bar{x}^{n-1}), \bar{x}^{n-2})$. Without loss of generality, suppose that $d_i(v) = 1$.[12] Now suppose there exists another agent $j \neq i, k$ such that $v_j < T_j^\mu(v_{-j})$ implying that $x = v_j < T_j^\mu(T_i^\mu(\bar{x}^{n-1}), \bar{x}^{n-2})$. Therefore, by **(d)** and condition (1) proved above, $v_i = T_i^\mu(x + \epsilon, \bar{x}^{n-2})$ if $x + \epsilon < T_j^\mu(T_i^\mu(\bar{x}^{n-1}), \bar{x}^{n-2})$. Therefore, by Lemma 1 and Result 1, we get that:

$$T_i^\mu(x + \epsilon, \bar{x}^{n-2}) - T_i^\mu(\bar{x}^{n-1}) = \begin{cases} 0 & \text{for all } 0 \leq \epsilon < T_j^\mu(T_i^\mu(\bar{x}^{n-1}), \bar{x}^{n-2}) - x \\ \text{positive} & \text{for all } \epsilon > T_j^\mu(T_i^\mu(\bar{x}^{n-1}), \bar{x}^{n-2}) - x \end{cases} \tag{1}$$

Note that by Result 1, $T_i^\mu(.)$ values must *not* depend on the value reported agent i. On the other hand, equation must hold true for all values of $x \geq 0$. Now, consider the possibility that $T_j^\mu(.)$ is independent of i's reported value. This would imply that, at any profile $\hat{v}$ where $\hat{v}_j > T_j^\mu(\bar{0}^{n-2})$, $\hat{v}_i > T_i^\mu(\hat{v}_j, \bar{0}^{n-2})$, and $\hat{v}_l = 0$ for all $l \neq i, j$; the decision values $d_i(\hat{v}) = d_j(\hat{v}) = 1$, which contradicts a single object being allocated. Therefore, (1) implies that $T_i(\bar{x}^{n-1})$ is a constant for all values of $x \geq 0$, and all $i \in N$. In that case, we can define n non-negative finite real numbers $K_1, K_2, \ldots, K_n$ such that for any $l \in N$, $K_l = T_l^\mu(\bar{x}^{n-1}), \forall\, x \geq 0$. Now, given the finite number of agents, we can choose a $K^* > \max_{l \in N} K_l$, and consider the profile of reports v^* where every agent i reports the same value K^*. By construction, $K_i \geq T_i^\mu(v_{-i})$ for all i, and so, by Result 1, $d_i(v^*) = 1$ for all i which again contradicts the single object setting.

Hence, we can infer that, for all $j \neq i, k$, $v_j = T_j^\mu(T_i^\mu(\bar{x}^{n-1}), \bar{x}^{n-2})$ and so, the condition (2) follows. □

Lemma 3 *If a mechanism $\mu = (d, \tau) \in \Gamma$ satisfies SP then for all $x \geq 0$ and all $i \in N$,*

$$T_i^\mu(\bar{x}^{n-1}) = x$$

Proof: Fix any mechanism $\mu = (d, \tau) \in \Gamma$ that satisfies SP. Fix any value $x \geq 0$ and any agent $i \in N$. Consider the two possibilities: (i) $T_i^\mu(\bar{x}^{n-1}) < x$, and (ii) $T_i^\mu(\bar{x}^{n-1}) > x$.

[12] The only other possibility is that $d_k(v) = 1$. In that case too, the same arguments would lead to the same conclusions.

Consider the possibility (i). Applying condition (1) of Lemma 1 for profile $\bar{x}^n$, (**A**) $T_l^\mu(\bar{x}^{n-1}) > x$ for all $l \neq i$. Now fix any $j \neq k \neq i$. Further, applying Lemma 2 for profiles $\hat{v}$ and $\tilde{v}$, where $(\hat{v}_i, \hat{v}_{-i}) = (T_i^\mu(\bar{x}^{n-1}), \bar{x}^{n-1})$ and $(\tilde{v}_k, \tilde{v}_{-k}) = (T_k^\mu(\bar{x}^{n-1}), \bar{x}^{n-1})$, respectively; we get that (**B**) $x = T_j^\mu(\hat{v}_{-j}) = T_j^\mu(\tilde{v}_{-j})$. Now, by Lemma 2, **(A)** and **(B)**, $x = T_j^\mu(\hat{v}_{-j}) \leq T_j^\mu(\bar{x}^{n-1}) \leq T_j^\mu(\tilde{v}_{-j}) = x$, which establishes that $T_j^\mu(\bar{x}^{n-1}) = x$, which contradicts (**A**). Hence, possibility (i) cannot hold.

For possibility (ii), consider the profile $\bar{x}^n$, and note that, by Result 1, $d_i(\bar{x}^n) = 0$. So, there exists a $j \neq i$ such that $d_j(\bar{x}^n) = 1$. Now, if $x > T_j^\mu(\bar{x}^{n-1})$, then arguing as above, we can show that there exists some $l \neq j$ such that $x = T_l^\mu(\bar{x}^{n-1})$, which would contradict Lemma 1. Now if $x = T_j^\mu(\bar{x}^{n-1})$, then by applying Lemma 2 to the profile $\bar{x}^n$, we get that $x = T_i^\mu(\bar{x}^{n-1})$, which contradicts the possibility (ii). Hence, the result follows. □

Lemma 4 *If a mechanism $\mu = (d, \tau) \in \Gamma$ satisfies SP then for all $i \in N$, and for all $v \in \mathbb{R}_+^N$,*

$$T_i^\mu(v_{-i}) = \max_{j \neq i} v_j$$

Proof: Fix any mechanism $\mu = (d, \tau) \in \Gamma$ that satisfies SP. Also, fix any agent $i \in N$, and any $z \in \mathbb{R}^{n-1}$. Without loss of generality, assume that $z = (z^1, z^2, \ldots, z^{n-1})$ where $z^k \geq z^{k+1}$ for all $k = 1, \ldots, n-2$. Therefore, we need to show that $T_i^\mu(z) = z^1$. For the sake of notational simplicity, let $\theta := z^1$.

Now, fix any $\epsilon > 0$ and consider the profiles v^ϵ and $v^{-\epsilon}$ such that $v_i^\epsilon = \theta + \epsilon, v_{-i}^\epsilon = \bar{\theta}^{n-1}$ and $v_i^{-\epsilon} = \theta - \epsilon, v_{-i}^{-\epsilon} = \bar{\theta}^{n-1}$. By Lemma 3, $T_i^\mu(\bar{\theta}^{n-1}) = \theta$, and so, by construction, $v_i^\epsilon > T_i^\mu(\bar{\theta}^{n-1})$ and $v_i^{-\epsilon} < T_i^\mu(\bar{\theta}^{n-1})$. Now, by condition (1) of Lemma 2 and construction of θ, $T_i^\mu(\bar{\theta}^{n-1}) \geq T_i^\mu(\bar{\theta}^{n-2}, z_n) \geq \ldots \geq T_i^\mu(z)$ implying that $v_i^\epsilon > T_i^\mu(z)$. Arguing similarly for profile $v^{-\epsilon}$, we get that $v_i^{-\epsilon} < T_i^\mu(z)$. Thus, we get that for all $\epsilon > 0$,

$$v_i^{-\epsilon} < T_i^\mu(z) < v_i^\epsilon$$

which implies that $T_i^\mu(z) = \theta = z^1$. Hence the result follows. □

It easy to see that the threshold function specified in Lemma 4 requires the object to be allotted to the highest bidders at all valuation profiles, and hence, describes an efficient mechanism.

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