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Characterization of Vickrey auction with reserve price for multiple objects

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Abstract

This paper completely characterizes Vickrey auction with reserve price [VARP], in single and multiple objects settings, using normative and strategic axioms. In particular, it provides a topological interpretation of reserve price as the *infimum* of a particular set of non-negative real numbers.

In the single object case, we find that a strategyproof mechanism satisfies anonymity in welfare, agent sovereignty and non-bossiness in decision if and only if it has a VARP allocation rule. To extend this result to the multiple objects setting, we introduce a continuity condition and show that any continuous and strategyproof mechanism satisfies the aforementioned properties (and a mild regularity condition) if and only if it has a VARP allocation rule.

JEL classification: C72; C78; D71; D63

Keywords: Anonymity in welfare, agent sovereignty, non-bossiness in decision, continuity, strategyproof mechanism

1 Introduction

It is well known that reserve pricing at auctions is an important method of ensuring that the seller revenue is not too low (Ausubel and Cramton [3]). Vickrey auctions, on other hand, ensure that the objects are allocated efficiently and that agents have no incentive to misreport irrespective of what other agents are reporting. Therefore, Vickrey auction with reserve price [VARP][1] is a useful mechanism for accomplishing both objectives of efficient allocation of objects and avoidance of low seller revenues. It is, therefore, no

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[1]Vickrey auction with reserve price is a mechanism with a special allocation rule where objects are allocated only to agents whose bids are not less than the reserve price. Further, winners of object pay the maximum of the reserve price and the greatest losing bid as price and non-winners pay nothing.

surprise that auctioneers have been recorded to be using VARP as early as 1897.[2] While revenue generation properties of VARP have been well documented over time, there is a dearth of literature on ethical properties of VARP. This is in contrast to a large literature providing normative characterizations of Vickrey auction without reserve prices.[3] This paper attempts to fill this gap by completely characterizing VARP, both in single and multiple objects settings (with unit demand), using normative axioms.

We present the idea of ethical mechanisms by invoking two popular notions of fairness: *anonymity in welfare* and *agent sovereignty*.[4] A mechanism is said to satisfy anonymity in welfare if utility levels of any two agents get interchanged, when their valuations are interchanged with all other agents' valuations remaining unchanged. A mechanism is said to satisfy agent sovereignty if it provides each agent with some opportunity to get an object, irrespective of what the other agents are reporting.

Further, we describe the idea of mechanisms being immune to manipulation by invoking the concept of *strategyproofness*. A mechanism is said to be strategyproof if truth-telling is a weakly dominant strategy for all agents in the direct revelation game induced by it.

We use an additional axiom of non-bossiness which requires that no agent be able to affect the allocation decision of another agent without affecting her own allocation decision. Since this is a different version of the conventional non-bossiness axiom of Satterthwaite and Sonnenschein [26], we call it *non-bossiness in decision*.[5] As argued by Thomson [29], non-bossiness of decision, in company of strategyproofness, embodies strategic restrictions that discourage collusive practices where agents form groups to misreport in a manner that changes the allotment decision to benefit one member of the group while not making any other member worse off.

In the single object case, we show that a strategyproof mechanism satisfies anonymity, agent sovereignty and non-bossiness in decision only if it has an allocation rule same as that of a VARP.[6] Then we completely characterize the class of mechanisms that satisfy

[2]Lucking-Reiley [12] note that VARP was found to be held as early as 1897 by pioneering stamp dealer William P. Brown of New York. It must, however, be noted that in spite of its long history VARP has not been a popular mode of auction over time (see Asubel and Milgrom [4]).

[3]Moulin [15], Ashlagi and Serizawa [1], Sakai [22], [24], Saitoh and Serizawa [21], Mukherjee [18] etc.

[4]These axioms have also been used in related literature. In particular, anonymity in welfare has been used by Ashlagi and Serizawa [1], Hashimoto and Saitoh [7]; agent sovereignty has been used by Marchant and Mishra [13], Moulin and Shenker [17].

[5]Similar notions of non-bossiness have been used by Satterthwaite and Sonnenschein [26], Svensson [28], Goswami, Mitra and Sen [6] etc. Note that our version of 'non-bossiness in decision' does not impose any restriction on transfers of a mechanism unlike the conventional version of non-bossiness used by Satterthwaite and Sonnenschein [26]. This version has also been used by Mishra and Quadir [14]. In fact, we show in the appendix (please refer to subsection 7.4, page 26) that any strategyproof mechanism violating non-bossiness in decision, would also violate the conventional Satterthwaite and Sonnenschein [24] non-bossiness.

[6]In a single object setting where objects are never left unallocated at any profile, Ashlagi and Serizawa [1] show that any mechanism satisfying anonymity, strategyproofness and individual rationality must have a Vickrey auction allocation rule without any reserve price. Mukherjee [18] strengthens this

anonymity, agent sovereignty, strategyproofness and non-bossiness in decision. Any mechanism in this class satisfies a mild *zero-utility* condition (requiring that any agent with zero valuation for the object should get zero utility by participating in the mechanism), if and only if it is a VARP.

Unfortunately, these characterizations fail to hold in the multiple homogeneous objects case straightaway. That is because with multiple objects, any number of objects may be withheld by the planner leading to a proliferation of the number of possible decisions at any valuation profile. For example, when there are three objects to be allocated; at any valuation profile, the planner must choose from four possible decisions of allocating $k \in \{0, 1, 2, 3\}$ objects. In contrast, with a single object to allocate; at any valuation profile, the planner has only two possible choices of either allocating the object or not. To address the subsequent technical complexities, we introduce a continuity condition, and show that any *continuous* mechanism satisfies anonymity, agent sovereignty, non-bossiness in decision, strategyproofness and zero-utility (and a mild regularity condition); if and only if it is a VARP. Thus, our paper completely characterizes the class of VARP in both single object and *multiple* objects settings.

1.1 Relation to literature

Perhaps the most popular paper on reserve pricing is Myerson [19]. Myerson [19], in an independent private value setting for a single indivisible object, identifies a particular VARP as *one* of the (Bayes-Nash incentive compatible) revenue maximizing mechanisms under the assumptions of: (i) symmetric bidders, (ii) distribution of valuations satisfying a regularity condition and (iii) the planner knowing this distribution with certainty. Further, Myerson [19] obtains a revenue maximizing mechanism involving different reserve prices for different agents if assumption (i) is violated. In contrast, for the single object case, our paper uses the same independent private value setting, without making the assumption (i) or any other distributional assumption, to show that any mechanism is an ethical (anonymous, agent sovereign and non-bossy) and strategyproof mechanism, if and only if it is a VARP. Thus, our result provides an interpretation of VARP (and hence, use of single *identical* reserve price across all bidders) even when bidder valuations are not symmetrically distributed. Additionally, unlike any other paper that we are aware of, our paper presents a characterization of VARP for multiple objects.

Some other papers, particularly relevant to our analysis are, Mishra and Quadir [14], Sakai [23], Klaus and Nichifor [10], and Tierney [30]. Mishra and Quadir [14] focus only on the single object allocation problem with money, and characterize the class of strat-

result by showing that any anonymous and strategyproof mechanism, in such a setting, must have a Vickrey auction allocation rule without reserve price. Mukherjee [18] further shows that this result continues to hold with multiple homogeneous objects, provided no object is left unallocated at any valuation profile.

egyproof and non-bossy (in decision) allocation rules. They show that for any reported valuation: the utility vector generated by the chosen allocation must be consistent to maximization of some '*monotone*' binary relation on the set of all possible utility vectors that may be realized. They also identify a continuity condition (in terms of allocations generated by a rule), and show that all strategyproof and non-bossy allocation rules which satisfy this condition, must consider a non-decreasing transformation of the reported valuation numbers, to allocate the single object to the agent who has the highest transformed valuation. In the first section of our paper, which deals with single object, unlike them, we characterize the exact Vickrey auction rule with reserve price (instead of large class of allocation rules), and identify the reserve price as an infimum of a specially constructed set.

Sakai [23], too, focuses only on the single object allocation problem, and completely characterizes the VARP mechanism as the only non-trivial mechanism that satisfies a 'weak' version of efficiency, strategyproofness and non-imposition. They also present some relations between equity and efficiency axioms, and use them to obtain other characterizations of VARP mechanism. However, all his characterizations require use of some "*parts of efficiency*" as exogenously imposed restrictions. In contrast, all efficiency properties of our results emanate from the interaction of axioms that are motivated by perspectives unrelated to efficiency.[7]

Klaus and Nichifor [10], too, focus on only the single object allocation problem, and provide a normative exposition of reserve prices without using any efficiency axiom. In their paper, too, reserve prices appear as implication of suitable chosen axioms (which are different to ours). However, their main characterization result presents a novel "*serial dictatorship with reservation prices*" mechanism, which involves *different* reserve prices for different agents arranged in a sequential order. In contrast, our paper focuses on mechanisms that are anonymous in welfare, which eliminates the possibility of different agents being treated as per different reserve prices.

Tierney [30] analyzes pairwise (weak) group strategyproof and continuous (in utility space) mechanisms that are anonymous in welfare for allocation of heterogeneous objects with unit demand in a quasi-linear environment, and characterizes a class of rules that assign separate reserve prices to 'real' objects, and possibly multiple reserve prices to the 'null' object (which denotes "*consuming none of the real objects*").[8] This result is difficult to motivate in our setting of homogeneous objects; (i) where agents never derive disutility from an object, and (ii) getting no object (that is, getting null object) is merely

[7]Sakai [23] shows in a single object setting that any mechanism satisfying 'weak efficiency', strategyproofness, and 'non-imposition', is either a VARP mechanism or a trivial mechanism that never allocates the object (or charges any price). While we use a zero-utility axiom that is logically equivalent to the 'non-imposition' property of Sakai [23]; our single object characterization is *logically independent* of all his characterizations, as, we use no axioms of efficiency, in "part" or otherwise.

[8]See fourth paragraph of page 6 in the working paper Tierney [30].

an implication of failing to meet the *single* reserve price for the real objects.[9] Further, the mechanisms characterized by Tierney [30], when reduced to single object setting entail a separate (possibly positive) reserve price for getting no object, which is contrary to our findings. Hence, our results are of independent interest to theirs. Finally, instead of treating reserve prices as a parameter, we present a topological interpretation of reserve price where it gets endogenously determined as an infimum of a special set of real numbers that follow from our axioms.

From a purely strategic perspective (without any normative axiomatic structure), a few notable recent works on reserve prices and their welfare and revenue effects are: Hu, Matthews and Zu [8], Kotowski [11], and Sano [25]. Unlike our paper, all these papers adopt the strategic perspective of Bayes' Nash incentive compatibility, under some chosen prior distribution of private informations.

The paper proceeds as follows. Section 2 presents the model and definitions. Section 3 presents the results on single and multiple objects. Section 4 discusses the independence of axioms. Section 5 discusses weakening of some of the axioms. Finally, Section 6 contains some of our concluding remarks. Proofs are relegated to the appendix (Section 7).

2 Model

We consider a situation where m homogeneous indivisible objects are available to be allocated to agents in $N = \{1, 2, \ldots, n\}$ with unit demand and the restriction $1 \leq m \leq n-1$. Each agent $i \in N$ has an independent private valuation $v_i \in \mathbb{R}_+$. For any $i \in N$, a generic allocation of i is denoted by (d_i, t) where d_i represents the object allocation decision taking values in $\{0, 1\}$ with $d_i = 1$ if and only if i gets an object, and t represents an amount of money. We assume that agents have quasilinear preferences over object and money, that is, utility to i from the allocation (d_i, t) is $d_i v_i + t$.

A mechanism is a tuple of functions (d^m, τ^m) such that at any reported profile of valuations $v \in \mathbb{R}_+^N$, each agent i is allocated a monetary transfer $\tau_i^m(v) \in \mathbb{R}$ and a decision $d_i^m(v) \in \{0, 1\}$. For any reported valuation profile $v \in \mathbb{R}_+^N$, define $W^m(v) := \{i \in N | d_i^m(v) = 1\}$ to be the set of agents that are allocated an object. Note that at any reported profile of valuations $v \in \mathbb{R}_+^N$, $|W^m(v)| \leq m$, that is, all objects need not get allocated at all reported profiles. Also define $B_0^m := \{v \in \mathbb{R}_+^N : W^m(v) = \emptyset\}$ to be the set of profiles at which no object is allocated. Therefore, the utility to any agent i with a true valuation of v_i at any reported profile $v' \in \mathbb{R}_+^N$, from the mechanism (d^m, τ^m) is given by $u((d_i^m(v'), \tau_i^m(v')); v_i) = v_i d_i^m(v') + \tau_i^m(v')$.

Let $\forall \, \{i, j\} \subseteq N$, $\forall \, v \in \mathbb{R}_+^N$, $v_{-i} := (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ and $v_{-i-j} := (v_{-i})_{-j}$.

[9] Given identical copies of an object being allocated, any agent expecting disutility from consuming one would simply choose to not participate in the allocation exercise.

Further, for all $r = 1, 2 \ldots, n$, define $v(r)$ to be the rth ranked valuation in a non-increasing arrangement of coordinates of any $v \in \mathbb{R}_+^N$. In case of ties while arranging the coordinates in such manner, without loss of generality, we use the tie-breaking rule $1 \succ \ldots \succ n$.[10] Finally, define for any $x \geq 0$, $\bar{x}^t := (x, x, \ldots, x) \in \mathbb{R}_+^t$ for all $t = 1, 2, \ldots, n$. Therefore, $\bar{x}^n = (x, x, \ldots, x) \in \mathbb{R}_+^n$ and $\bar{x}^{n-1} = (x, x, \ldots, x) \in \mathbb{R}_+^{n-1}$.

The following definition states Vickrey auction with reserve price $r \geq 0$ to be a mechanism with an allocation rule that gives out objects to the top (not more than m) bidders that have bid amounts in excess of r; and charges a price, only to the winners, that is equal to the greater value among r and the $(m+1)$th highest bid.

Definition 1. Any mechanism (d^{m^r}, τ^{m^r}) is said to be a *Vickrey auction with reserve price* $r \geq 0$ (VARP) if for all $i \in N$ and all $v \in \mathbb{R}_+^N$,

- $v_i < \max\{v_{-i}(m), r\} \implies d_i^{m^r}(v) = 0$
- $v_i > \max\{v_{-i}(m), r\} \implies d_i^{m^r}(v) = 1$
- $\tau_i^{m^r}(v) = \begin{cases} 0 & \text{if } d_i^{m^r}(v) = 0 \\ -\max\{v_{-i}(m), r\} & \text{if } d_i^{m^r}(v) = 1 \end{cases}$

Define $\Gamma^m := \{d^{m^r}, \tau^{m^r}\}_{r \geq 0}$ to be the class of VARP mechanisms.

A popular strategic axiom in independent private values setting, strategyproofness, eliminates the incentive to misreport valuation for each agent. It is defined as follows.

Definition 2. A mechanism (d^m, τ^m) satisfies *strategyproofness* (SP) if $\forall\, i \in N$, $\forall\, v_i, v_i' \in \mathbb{R}_+$, $\forall\, v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$,

$$u(d_i^m(v_i, v_{-i}), \tau_i^m(v_i, v_{-i}); v_i) \geq u(d_i^m(v_i', v_{-i}), \tau_i^m(v_i', v_{-i}); v_i)$$

Therefore, a strategyproof mechanism ensures that revealing the true valuation is a weakly dominant strategy for each agent in the ensuing game.

The following axiom states the idea of 'non-bossiness in decision' which requires that no agent be able to affect allocation decision of another agent without affecting her own allocation decision.

Definition 3. A mechanism (d^m, τ^m) satisfies *non-bossiness in decision* (NBD) if for all $i \in N$, all $v \in \mathbb{R}_+^N$ and all $v_i' \in \mathbb{R}_+$,

$$d_i^m(v) = d_i^m(v_i', v_{-i}) \implies d_j^m(v) = d_j^m(v_i', v_{-i}), \forall\, j \neq i$$

[10] For any $i \neq j$, $i \succ j$ means that the tie is broken in favour of agent i. That is, for any v, if $v_3 = v_7 > v_i$ for all $i \neq 3, 7$ and $3 \succ 7$, then $v(1) = v_3$.

As mentioned earlier, NBD embodies a strategic barrier to collusive practices where agents form groups to misreport in a manner that changes the allotment decision to benefit any one member of the group while not making any other member worse off.[11]

The following two definitions pertain to two different notions of fairness. They describe ethically desirable behaviour that a mechanism should exhibit in an idealized state of nature where there is no private information (that is, planner knows every agent's true valuation). The first definition states the fairness concept of anonymity in welfare which requires that utility derived from an allocation by any agent be independent of her identity. The second definition states the fairness idea that each agent should have an opportunity to get an object, irrespective of what the other agents are reporting.[12]

Definition 4. A mechanism (d^m, τ^m) satisfies *anonymity in welfare* (AN) if for all $i \in N$, all $v \in \mathbb{R}_+^N$ and all bijections $\pi : N \mapsto N$,

$$u(d_i(v), \tau_i(v); v_i) = u(d_{\pi i}(\pi v), \tau_{\pi i}(\pi v); \pi v_{\pi i})$$

where $\pi v := \left(v_{\pi^{-1}(k)}\right)_{k=1}^n$.

Definition 5. A mechanism (d^m, τ^m) satisfies *agent sovereignty* (AS) if for all $i \in N$ and all $v \in \mathbb{R}_+^N$, there exists $v_i' \in \mathbb{R}_+$ such that

$$d_i^m(v_i', v_{-i}) = 1$$

Finally, the following axiom implies the fairness perception that if an agent has zero valuation for the object, then the agent must not get a positive or negative utility by merely participating in the mechanism.

Definition 6. A mechanism (d^m, τ^m) satisfies *zero-utility* if for all $i \in N$ and all $v_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$,

$$u(d_i^m(0, v_{-i}), \tau_i^m(0, v_{-i}); 0) = 0.$$

Note that for our single object setting, this zero-utility condition is logically equivalent to the *non-imposition* condition of Sakai [23].[13]

[11]See Thomson [29]. Also to see the kind of undesirable mechanisms that NBD excludes, consider the following example. For any profile v: (i) if there exists an agent i such that $v_i \in [0, b(1))$ and v_i is an irrational number, then for all $j \in N$, $(d_j^m(v), \tau_j^m(v)) = (0, 0)$, and (ii) if not, then the allocations are made according to a VARP with reserve price 10. Thus, NBD disallows an agent from generating consequences that do not affect her own self, but affect the allocation decisions of other agents.

[12]Moulin [16] mentions the agent sovereignty axiom to be"*reminiscent of the citizen sovereignty of classical social choice.*"

[13]The latter requires that $\tau_i^1(0, v_{-i}) = 0$ for all i and $v_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$.

3 Main results

For the sake of simplicity of notation, henceforth, we suppress the superscript m while describing a mechanism (d^m, τ^m) whenever the number of objects being allocated is clear from the ambient context.

We begin by noting the following well known result which establishes that the decision rule implicit in any strategyproof mechanism must be non-decreasing in one's own reported value.[14] In particular, for any agent i and any profile of valuations v_{-i}, there must exist a threshold price $T_i(v_{-i})$ such that: i gets an object if v_i *strictly* exceeds $T_i(v_{-i})$ and fails to get an object if v_i is *strictly* less than $T_i(v_{-i})$. Further, if a strategyproof mechanism satisfies AS, then these threshold prices must be finite. Finally, SP and AS imply that the transfer of agent i when getting the object, must exceed that when not getting the object, by $T_i(v_{-i})$.[15]

Fact 1. Any mechanism (d, τ) satisfies SP and AS, if and only if $\forall\, i \in N$ and $\forall\, v_{-i} \in \mathbb{R}_+^{N\setminus\{i\}}$, there exist real valued functions $K_i : \mathbb{R}_+^{N\setminus\{i\}} \mapsto \mathbb{R}$ and $T_i : \mathbb{R}_+^{N\setminus\{i\}} \mapsto \mathbb{R}$ such that

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > T_i(v_{-i}) \\ 0 & \text{if } v_i < T_i(v_{-i}) \end{cases} \quad \text{and} \quad \tau_i(v) = \begin{cases} K_i(v_{-i}) - T_i(v_{-i}) & \text{if } d_i(v) = 1 \\ K_i(v_{-i}) & \text{if } d_i(v) = 0 \end{cases}$$

Remark 1. Note that Fact 1 allows for arbitrary tie breaking in strategyproof mechanisms, for valuation profiles $v \in \mathbb{R}_+^N$ such that $v_i = T_i(v_{-i})$ for some $i \in N$. Thus, Fact 1 establishes that for any valuation profile v, $d_i(v) = 1$ implies that $v_i \geq T_i(v_{-i})$ and $d_i(v) = 0$ implies that $v_i \leq T_i(v_{-i})$. In this paper, without loss of generality, we use the following tie breaking rule:

> *For any profile v, define $X(v) := \{i \in N : v_i > T_i(v_{-i})\}$, and $Y(v) := \{i \in N : v_i = T_i(v_{-i})\}$. At any profile v, if $|Y(v)| \leq m - |X(v)|$ then all agents in $Y(v)$ are allocated an object each, or else the top $m - |X(v)|$ agents in $Y(v)$ according to the order $1 \succ 2 \succ \ldots \succ n$ are allocated an object each.*[16]

3.1 Single object: $m = 1$

In this section we study the single object case. The following proposition states that any mechanism satisfying AN, AS, NBD and SP, must have an allocation rule same as that of a VARP. That is, any such mechanism must have an associated reserve price that is *common* across all agents.

[14] This result can also be found as Proposition 9.27 in Nisan [20] and Lemma 1 in Mukherjee [18]

[15] AS ensures that the threshold functions have finite images.

[16] In light of Fact 1, at any valuation profile v, a tie-breaking rule is invoked *only if* there exists an agent i such that $v_i = T_i(v_{-i})$. And so, no matter how this tie is broken, the transfer rule of Fact 1 implies that i is indifferent between getting and not getting the object (as her utility is $K_i(v_{-i})$ in both cases).

Theorem 1. A mechanism (d, τ) satisfies properties AN, AS, NBD and SP *only if* $\exists\, r \geq 0$ such that for all $i \in N$ and all $v \in \mathbb{R}^N_+$,

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > \max\{v_{-i}(1), r\} \\ 0 & \text{if } v_i < \max\{v_{-i}(1), r\} \end{cases}$$

Proof: We accomplish this proof in three stages.[17] First, in Lemma 2 of Appendix, we establish existence of a real η which is well defined with respect to a set of valuations where at least one object is allocated. Then, in subsection 7.2 of Appendix, we show that for all v and all i, (i) $v_i < \max\{v_{-i}(1), \eta\}$ implies $d_i(v) = 0$, and (ii) $v_i > \max\{v_{-i}(1), \eta\}$ implies that $d_i(v) = 1$. This allows us to establish existence of a reserve price $r := \eta$ such that $T_i(v_{-i}) = \max\{v_{-i}(1), r\}$ for all v and all i. □

Remark 2. Kazamura, Mishra and Serizawa [9], henceforth, referred to as KMS, show that any mechanism satisfying AN, SP and 'loser payment independence' (requiring that loser at any profile pay the same amount irrespective of her preference for the object), must be an *adjusted Vickrey auction with a variable reserve price.* Theorem 1 complements this result by showing that: in a quasilinear setting, any mechanism satisfying AN, AS, SP and NBD, must have an allocation rule same as that of a VARP (that is, uses a common reserve price).

Note that, for the single object case, our tie breaking rule implies that for any v with $v_i \leq T_i(v_{-i}), \forall i \in N$, the object is allocated to the top most agent in $Y(v)$ according to the order $1 \succ 2 \succ \ldots \succ n$. Therefore, Theorem 1 provides a novel topological interpretation to the reserve price value of a VARP. That is, it establishes that the reserve price used in a VARP mechanism, must also be the *infimum* of a set S consisting of non-negative real numbers satisfying the following property: if all agents bid the same number from S, then at least one object is allocated. As we shall see later, this interpretation continues to hold (in Proposition 3) when there are more than one objects to allocate. This idea is expressed in the following corollary.

Corollary 1. For any mechanism $(d^{1^r}, \tau^{1^r}) \in \Gamma^1$,

$$r = \inf\{x \geq 0 : \bar{x}^n \notin B_0^1\}$$

Proof: It is easy to that any VARP satisfies AN, AS, NBD, and SP. Hence, from proof of Theorem 1, the result follows. □

Next, we define a special class of mechanisms that employ uniform reserve prices in their allocation and transfer rules.

[17] See Appendix for full details.

Definition 7. Let $\mathcal{M}^1$ be the class of mechanisms (d,τ) such that for all $i \in N$ and all $v \in \mathbb{R}^N_+$,

- $d_i(v) = \begin{cases} 1 & \text{if } v_i > \max\{v_{-i}(1), r\} \\ 0 & \text{if } v_i < \max\{v_{-i}(1), r\} \end{cases}$

- $\tau_i(v) = \begin{cases} K(v_{-i}) - \max\{v_{-i}(1), r\} & \text{if } d_i^r(v) = 1 \\ K(v_{-i}) & \text{if } d_i^r(v) = 0 \end{cases}$

where $K : \mathbb{R}^{n-1}_+ \mapsto \mathbb{R}$ is a *symmetric* function.[18]

Thus, $\mathcal{M}^1$ is a special class of mechanisms with the VARP allocation rule. It contains an interesting sub-class of mechanisms with this allocation rule but not the VARP transfer rule. This is the class of *maxmed* mechanisms introduced by Sprumont [27]. These mechanisms belong to $\mathcal{M}^1$ and can be obtained by setting

$$K(v_{-i}) = med\left\{0, v_{-i}(1) - r, \frac{r}{n-1}\right\}, \forall\, v \in \mathbb{R}^n_+, \forall\, i \in N,$$

where for any three real numbers x, y, z, $med\{x, y, z\}$ denotes the median on the three numbers.

The following theorem completely characterizes $\mathcal{M}^1$.

Theorem 2. Any mechanism (d,τ) satisfies AN, AS, NBD and SP *if and only if* $(d,\tau) \in \mathcal{M}^1$.

Proof: See Appendix. □

Note that all mechanisms in $\mathcal{M}^1$ which have a special $K(.)$ function such that $K(z) = 0$ for all $z \in \mathbb{R}^{n-1}_+$, are VARP. That is, the class of VARP mechanisms for single object; Γ^1 is a subset of $\mathcal{M}^1$, i.e., $\Gamma^1 \subseteq \mathcal{M}^1$. We use this relation to obtain the following corollary which completely characterizes Γ.

Corollary 2. A mechanism (d,τ) satisfies AN, AS, NBD, SP and zero-utility *if and only if* $(d,\tau) \in \Gamma^1$.

Proof: The proof of sufficiency is easy to check. To see the necessity, fix any $i \in N$ and any $v_{-i} \in \mathbb{R}^{n-1}_+$. Consider the profile $(0, v_{-i})$. From Theorem 1 and zero-utility condition, it follows that $u_i(d_i(0, v_{-i}), \tau_i(0, v_{-i}); 0) = K(v_{-i}) = 0$. Hence, the result follows. □

Remark 3. Corollary 2 also follows from KMS. As mentioned earlier, they show in this discussion paper that in a single object setting with general (possibly non-quasilinear)

[18] A function of $k \in \mathbb{N}$ variables is said to be *symmetric* if the function value at any k-tuple of arguments is the same as the function value at any permutation of that k-tuple.

preferences, any mechanism satisfies AN, SP and loser payment independence, if and only if it is an *adjusted Vickrey auction with a variable reserve price.* In our setting: (i) SP and zero-utility condition imply the KMS axiom of loser payment independence, (ii) NBD rules out Vickrey mechanisms where the reserve price may depend on preference of other agents. Thus, their result implies our Corollary 2. Finally, dropping AS from the statement of Corollary 2 would lead to an additional trivial mechanism that never gives out the object and charges zero transfers.[19]

3.2 Multiple homogeneous objects: $m > 1$

In this section we study the case where number of objects/copies m can take any integer value from 2 to $n-1$. Ideally, the results in single object case should translate directly to the multiple homogeneous objects (with unit demand) setting. However, that is not the case. The reason for this are the following two complications that arise out of the multiple objects setting.

The first complication is that, at any profile, it no longer follows from any one agent getting an object, that other agents get no objects. Thus, the inherent externality of the single object setting, becomes very weak when $m > 1$. The second complication is that the planner, at any valuation profile, can now choose to allocate any k out of the m available objects (where $k \in \{0, 1, \ldots, m\}$). Both these issues lead to a large expansion of the class of mechanisms that satisfy our basic axioms of anonymity, agent sovereignty, strategyproofness, and non-bossiness in decision. Thus, our basic axioms are no longer enough to obtain a characterization like Theorem 1 in the multiple objects setting. To arrive at such characterization result, we define the following two technical conditions:

Definition 8. A mechanism (d, τ) is said to be *continuous* if for any $\zeta \in \{0, 1\}$, any $i \in N$ and any sequence of profiles $\{v^k\}$ that converges to $\tilde{v}$, whenever $d_i(v^k) = \zeta$ for all k,

$$d_i(\tilde{v}) \neq \zeta \implies u((1, \tau_i(\tilde{v})); \tilde{v}_i) = u((0, \tau_i(\tilde{v})); \tilde{v}_i)$$

Definition 9. A mechanism is said to be *regular* if for all $m > m' \in \mathbb{N}$, $B_0^m \subseteq B_0^{m'}$.

A mechanism is *continuous* if it satisfies the property that: whenever the allocation decision of an agent i is not preserved in limit, the transfer assigned to i at the limit profile is such that she is indifferent between getting or not getting the object.[20] On the

[19]We thank an anonymous referee and the associate editor for pointing these logic out.

[20]For example, any VARP mechanism is continuous because at all profiles where all agents bid the same value, everyone gets 0 utility irrespective of winning an object or not. To see the kind of peculiar mechanisms ruled out by the restriction of continuity, consider w.l.o.g. a two object - three agent setting. Consider a mechanism in the class characterized by Fact 1 such that at all bid profiles, it allocates both objects to the first and second highest bidder whenever either of their bids is greater than or equal to 10, or else no objects are allocated. Further, any agent who is not allocated an object receives zero transfer, while any agent who is allocated an object pays a price equal to: 10 if bids of all other agents

other hand, the *regularity* of a mechanism requires that at any valuation profile: if no objects are allocated when $m \geq 2$ copies are available, then no objects must be allocated when $m-1$ copies are available. This property rules out strange mechanisms where abundance of objects leads to scarcity in allocations.[21]

The following theorem states that any continuous regular mechanism satisfying AN, AS, NBD and SP, must have an allocation rule same as that of a VARP (for m objects). That is, any such mechanism must have an associated reserve price that is *common* across all agents.

Theorem 3. A continuous regular mechanism (d, τ) satisfies properties AN, AS, NBD and SP *only if* $\exists\, r \geq 0$ such that for all $i \in N$ and all $v \in \mathbb{R}_+^N$,

$$d_i(v) = \begin{cases} 1 & \text{if } v_i > \max\{v_{-i}(m), r\} \\ 0 & \text{if } v_i < \max\{v_{-i}(m), r\} \end{cases}$$

Proof: The proof is presented in the subsection 7.4 in the Appendix. It consists of four steps. Briefly, it shows that for any arbitrary continuous regular mechanism (d, τ) that satisfies AN, AS, NBD, SP: (i) the associated threshold function $T(.)$ is continuous, (ii) at least one object is allocated at a profile $\bar{\eta}^n$ where $\eta := \inf\left\{x \geq 0 : 0 < \sum_{i \in N} d_i(\bar{x}^n) \leq m\right\}$, (iii) for any i and any v, $v_{-i}(1) = \eta \implies T(v_{-i}) = \eta$, and (iv) for any i and any v,

$$\begin{cases} T(v_{-i}) = \eta & v_{-i}(m) \leq \eta \\ T(v_{-i}) = v_{-i}(m) & \text{otherwise} \end{cases}$$

□

As mentioned earlier in the single object case, any VARP mechanism for allocation of $m > 1$ objects, must employ a uniform reserve price that is an *infimum* of special set of non-negative real numbers. This is described in below in the following corollary.[22]

Corollary 3. For any mechanism $(d^{m^r}, \tau^{m^r}) \in \Gamma^m$,

$$r = \inf\{x \geq 0 : \bar{x}^n \notin B_0^m\}$$

are strictly less than 10, or else the third highest bid. To see that this mechanism is discontinuous, consider a sequence of profiles $\{(10 - \frac{1}{k}, 9, 8)\}_k$. Note that for all k, the agent 2 does not get an object, but she gets an object at the limit profile $(10, 9, 8)$. However, 2 is charged a price 8 at the limit, which makes her prefer getting the object to not getting the object, that is, $u_2((1, -8); 9) > u_2((0, 0); 9)$.

[21]To further motivate this regularity condition, consider an auction house about to commit to certain rules of sale procedure for a period of time in future; during which this procedure may be invoked to allocate different batches (containing varying number) of similar or identical objects. This regularity condition rules out counter-intuitive sale procedures, where chance of winning an object for a bidder decreases when the number of available objects increases.

[22]Note that there may be mechanisms that employ VARP mechanism to allocate m objects even when there is a greater number m' of objects available to be allocated. However, we do not consider them as they do not conform to our tie-breaking rule (described in Remark 1).

Proof: It is easy to check that any VARP mechanism is continuous and satisfies AN, AS, NBD, SP. Hence, from proof of Theorem 3, the result follows. $\square$

Now, as in the single object case, we define special class of mechanisms $\mathcal{M}^m$, which employ reserve prices in their allocation of $m > 1$ objects and corresponding transfer rules.

Definition 10. Let $\mathcal{M}^m$ be the class of mechanisms (d, τ) such that for all $i \in N$ and all $v \in \mathbb{R}_+^N$,

- $d_i(v) = \begin{cases} 1 & \text{if } v_i > \max\{v_{-i}(m), r\} \\ 0 & \text{if } v_i < \max\{v_{-i}(m), r\} \end{cases}$

- $\tau_i(v) = \begin{cases} K(v_{-i}) - \max\{v_{-i}(m), r\} & \text{if } d_i(v) = 1 \\ K(v_{-i}) & \text{if } d_i(v) = 0 \end{cases}$

where $K : \mathbb{R}_+^{n-1} \mapsto \mathbb{R}$ is a *symmetric* function.

Similar to the single object case, we present the following proposition, which completely characterizes $\mathcal{M}^m$.

Theorem 4. Any continuous regular mechanism (d, τ) satisfies AN, AS, NBD and SP *if and only if* $(d, \tau) \in \mathcal{M}^m$.

Proof: See Appendix. $\square$

Again as observed in the single object case, all mechanisms in $\mathcal{M}^m$ which have a special $K(.)$ function such that $K(z) = 0$ for all $z \in \mathbb{R}_+^{n-1}$, are VARP. Hence, we get the following theorem which completely characterizes Γ^m.

Corollary 4. A continuous regular mechanism (d, τ) satisfies AN, AS, NBD, SP and zero-utility *if and only if* $(d, \tau) \in \Gamma^m$.

Proof: The proof of sufficiency is easy to check. To see the necessity, fix any $i \in N$ and any $v_{-i} \in \mathbb{R}_+^{n-1}$. Consider the profile $(0, v_{-i})$. From Theorem 4 and zero-utility condition, it follows that $u_i(d_i(0, v_{-i}), \tau_i(0, v_{-i}); 0) = K(v_{-i}) = 0$. Hence, the result follows. $\square$

4 Independence of Axioms

In this section we establish independence of axioms in our VARP characterization results. We begin with the single object case of Corollary 2 below, and the follow it up with the multiple object case of Corollary 4.

4.1 Corollary 2

This theorem uses the axioms of AN, AS, NBD, SP and zero-utility to characterize the class of VARP mechanism in the single object case. To show independence of axioms, we fix any axiom $P \in \{AN, AS, NBD, SP, \textit{zero-utility}\}$, and construct a mechanism that satisfies all of the aforementioned axioms other than P in the following manner:

(i) SP To see an example of mechanism that satisfies AN, AS, NBD and zero-utility, but not SP: consider the standard First Price Auction (FPA) where the highest bidder is allocated the object and is asked to pay her own bid as price, while all other agents pay (or receive) no money. It is well known that bidding true valuation is not a weakly dominant strategy at FPA, and so, it violates SP.

Note that this mechanism satisfies zero-utility as any agent with zero valuation, will either not get the object or get it at zero price. It also satisfies AS as every agent can theoretically get the object by outbidding others. It satisfies NBD because no agent change her bid in a manner such that her own object allocation decision remains unchanged but some else' allocation decision changes. Finally, note that for an FPA, at any valuation profile, utilities of any two agents would get interchanged if their valuations are interchanged. As mentioned in proof of Proposition 1, this feature is well known to imply AN as defined in Definition 4, and hence, we can see that FPA satisfies AN.

(ii) Zero-utility Consider a mechanism (d, τ) belonging to the class described by Fact 1. Suppose that for all $z \in \mathbb{R}_+^{n-1}$,

$$K(z) = \bar{K} > 0 \text{ and } T(z) = z(1)$$

By Fact 1, (d, τ) satisfies SP and AS. Further, note that (d, τ) is essentially a Vickrey auction with a peculiar modification where agents get a positive monetary transfer $\bar{K}$ irrespective of being allocated the object or not. Hence, it does not satisfy zero-utility. Further, since an agent gets the object only if she is the highest bidder, it is easy to see that (d, τ) satisfies NBD. Finally, it is easy to see that at any valuation profile, interchanging valuations of any two agents would interchange their utilities, and so, (d, τ) satisfies AN.

(iii) AS Consider a mechanism which never allocates any good at any profile, and always pays zero transfers to agents. It trivially satisfies SP, AN, NBD, and zero-utility. However, it violates AS because at all valuation profiles, each agent is consigned to not getting the object irrespective of what value others report.

(iv) AN Consider a mechanism (d, τ) belonging to the class described by Fact 1 for two agents $\{1,2\}$. Suppose that for all $x \geq 0$,

$$K_1(x) = x,\ K_2(x) = \max\{0, x-\epsilon\},\ \text{and}\ T_1(x) = x+\epsilon,\ T_2(x) = \max\{0, x-\epsilon\}$$

where $\epsilon > 0$. Recall that, as argued in section 5, (d,τ) satisfies NBD, SP, AS and zero-utility. By Proposition 1, (d,τ) does not satisfy AN as the $T_i(.)$ functions depend on agent identities.

(v) NBD Consider a mechanism (d,τ) belonging to the class described by Fact 1 such that for all $i \in N$ and $z \in \mathbb{R}_+^{n-1}$

$$K_i(z) = 0 \text{ and } T_i(z) = \begin{cases} \max\{z(1), 20\} & \text{if } z(1) > 10 \\ z(1) & \text{if } z(1) \leq 10 \end{cases}$$

Note that by Fact 1, (d,τ) satisfies SP and AS. Also, it can be easily seen that it satisfies zero-utility. Further, at any valuation profile, if valuations of any pair of agents are changed, their utilities get interchanged. Hence, (d,τ) satisfies AN. To see that this mechanism violates NBD, consider a three agent setting where the valuation profile $(v_1, v_2, v_3) = (15, 8, 7)$. Note that according our decision rule $d(15,8,7) = (1,0,0)$. But if agent 2 unilaterally changes her reported valuation to 11, the decision changes to $d(15,11,7) = (0,0,0)$, where agent 2 continues to not get the object but agent 1's decision outcome is affected as she no longer gets the object. Thus, (d,τ) does not satisfy NBD.

4.2 Corollary 4

Note that a major part of the complexity of analysis in this paper, particularly in the multiple objects case, arises from the fact that *all* objects may not be allocated at all valuation profiles. However, this feature also allows us to use some of the aforementioned mechanisms (used for the single object case) to show independence of axioms in Corollary 4. Recall that Corollary 4 invoked seven axioms: continuity, regularity, AN, AS, NBD, SP and zero-utility. We show independence between them in the following manner:

(i) SP As before, we consider the FPA, where the highest bidder gets an object and pays her own bid her price, while all other agents get no objects or non-zero transfers, *irrespective of the objects available.* As argued earlier in the single object case, this mechanism violates SP. It can easily be checked that this mechanism satisfies AS, and regularity. Arguing as in the single object case, we can see that it satisfies AN and NBD.

To check that FPA satisfies continuity, we use the idealized state of nature mentioned earlier (needed to evaluate axioms) where we suppose that there is no private information. Now, fix any $\zeta \in \{0,1\}$, any agent $i \in N$, and consider any sequence $\{v^n\}$ such that (i) $v^n \geq 0$, (ii) converges to some $v' \geq 0$, and (iii) for all n, $d_i(v^n) = \zeta$. Now if $\zeta = 1$ and $d_i(v') \neq \zeta$, we can infer that i is indifferent in terms of utility between winning and losing the object at profile v'. That is because, by construction, when not getting the object i gets zero utility, which is same as what she would have got had she won the object and paid her own bid v'_i. Arguing similarly when $\zeta = 0$ and $d_i(v') \neq \zeta$, we can see that i is indifferent in terms of utility between winning and losing the object at profile v'. Hence, continuity of FPA follows.

(ii) AS As before, consider a mechanism which never allocates any good at any profile, and always pays zero transfers to agents. It trivially satisfies continuity, regularity AN, NBD, SP and zero-utility. However, it violates AS because at all valuation profiles, each agent is consigned to not getting the object irrespective of what value others report.

(iii) Zero-utility Consider a mechanism (d, τ) belonging to the class described by Fact 1. Suppose that there are $m > 1$ objects to be allocated, and for all $i \in N$, $z \in \mathbb{R}_+^{n-1}$,

$$K_i(z) = \bar{K} > 0 \text{ and } T_i(z) = z(m)$$

As argued in the single object case, this mechanism satisfies AN, SP and AS but does not satisfy zero-utility. Note that at any valuation profile, the top m valuation agents get the objects, and all objects are allocated. Thus, the winners cannot affect anybody else' allocation decision without changing their own, and the same goes for the losers. Hence, this mechanism satisfies NBD. Further, this mechanism trivially satisfies regularity since it allocates all objects at all profiles (irrespective of the number of objects). To see how this mechanism satisfies continuity, note that its threshold function is continuous. So, as earlier, we can fix any $\zeta \in \{0,1\}$, any agent $i \in N$, and consider any sequence $\{v^n\}$ such that (i) $v^n \geq 0$, (ii) converges to some $v' \geq 0$, and (iii) for all n, $d_i(v^n) = \zeta$. Now, if $\zeta = 0$, then $v_i^n \leq T_i(v_{-i}^n)$, for all n. Therefore, (ii) implies that $\{v_{-i}^n\}$ converges to v'_{-i}, and so, by continuity of $T_i(.)$, we get that in limit, $v'_i \leq T_i(v'_{-i})$ implying that either $v'_i < T_i(v'_{-i}) \implies d_i(v') = 0 = \zeta$ or else $v' = T_i(v'_{-i}) \implies u((1, \bar{K} - T_i(v'_{-i})); v'_i) = u((0, \bar{K}); v'_i) = \bar{K}$ (by Fact 1). Now, if $\zeta = 1$, arguing as above, we can infer that (ii) implies that $v'_i \geq T_i(v'_{-i})$, and so, either $v'_i > T_i(v'_{-i}) \implies d_i(v') = 1 = \zeta$ or else $v' = T_i(v'_{-i}) \implies u((1, \bar{K} - T_i(v'_{-i})); v'_i) = u((0, \bar{K}); v'_i) = \bar{K}$ (by Fact 1). Thus, continuity of the mechanism follows.

(iv) NBD Consider a mechanism (d, τ) belonging to the class described by Fact 1 such that for all $i \in N$, $z \in \mathbb{R}^{n-1}_+$,

$$K_i(z) = 0 \text{ and } T_i(z) = z(1) + z(n-1)$$

As before, this mechanism satisfies SP and AS and zero-utility. Since the $T(.)$ is continuous, arguing as above, we can show that (d, τ) satisfies continuity. Also, since $T(.)$ is independent of agent identity, arguing as earlier instances, we can infer that (d, τ) satisfies AN. Also, by construction: (i) (d, τ) allocates *at most one object* at any valuation profile irrespective of the value m takes in $\{2 \ldots, n-1\}$, and (ii) the $T(.)$ function does not depend on value of m. Hence, (i) and (ii) imply that (d, τ) satisfies regularity. Finally, to see that this mechanism does not satisfy NBD, consider a three agent setting, and two valuation profiles $(15, 10, 2)$ and $(15, 10, 6)$. Note that: $d(15, 10, 2) = (1, 0, 0) \neq (0, 0, 0) = d(15, 10, 6)$, and thus, as before, we have a violation of NBD.

(v) AN Consider a 2 object setting with 3 agents $\{1, 2, 3\}$. Consider the mechanism (d, τ) such that for all $z \in \mathbb{R}^2_+$,

$$T_i(z) = \begin{cases} z(1) + 1 & \text{if } i = 1 \\ \max\{0, z(1) - 1\} & \text{otherwise} \end{cases} \quad \text{and } K_i(z) = 0$$

As argued in the single object case, this mechanism violates AN but satisfies AS, NBD, SP, and zero-utility. It can easily be seen that the $T(.)$ function is continuous. Finally, as in the previous case, the $T_i(.)$ functions do not depend on whether $m = 1$ or $m = 2$, and so, (d, τ) satisfies regularity.

(vi) Regularity Consider a mechanism (d, τ) in a three agent setting such that for all $i \in N$, $z \in \mathbb{R}^{n-1}_+$,

$$K_i(z) = 0 \text{ and } T_i(z) = \begin{cases} \max\{z(1), 11\} & \text{if } m = 2 \\ z(1) & \text{if } m = 1 \end{cases}$$

To see that (d, τ) does not satisfy regularity: note that if there is one object to be allocated, then $d(10, 5, 2) = (1, 0, 0)$, and so, $(10, 5, 2) \notin B^1_0$. However, when there are two objects to be allocated, $d(10, 5, 2) = (0, 0, 0)$, that is, $(10, 5, 2) \in B^2_0$.

On the other hand, it is easy to see, using earlier arguments, that (d, τ) satisfies AN, AS, continuity, SP, and zero-utility.

(vii) Continuity Consider a mechanism (d, τ) in a 2 object-3 agent setting, such that for all $i \in N$, $z \in \mathbb{R}^{n-1}_+$,

$$K_i(z) = 0 \text{ and } T_i(z) = \begin{cases} 10 & \text{if } z(1) \in [0, 10) \\ z(m) & \text{if } z(1) \geq 10 \end{cases}$$

Note that, as argued in Step 1 of section 7.4, if a mechanism satisfying SP and AN satisfies continuity, then the associate threshold $T(.)$ function must also be continuous. Here, $T(.)$ can be seen to be not continuous as $\{T(10 - \frac{1}{n}, 5, 5)\} \to 10$ but $T(10, 5, 5) = 5$. Hence, (d, τ) is *not* continuous.

Further, as argued earlier, it can be easily seen that this mechanism satisfies AS, AN, NBD, SP and zero-utility. Also, note that any profile $v \in B^2_0$, it must be that $v(1) < 10$, which implies that $v \in B^1_0$. Hence, $B^2_0 \subseteq B^1_0$, and so, regularity is satisfied.

5 Discussion

In this section, we discuss the possibility of weakening the fairness axiom AN to the axiom of *equal treatment of equals*, which is defined as following:

Definition 11. A mechanism (d^m, τ^m) satisfies *equal treatment of equals* (ETE) if $\forall\, i \neq j \in N$, $\forall\, v \in \mathbb{R}^N_+$,

$$[v_i = v_j] \implies [u(d^m_i(v), \tau^m_i(v); v_i) = u(d^m_i(v), \tau^m_i(v); v_j)].$$

It is easy to see that AN implies ETE, but not the other way around.

Note that our characterization results Theorem 1 and Theorem 2 would fail to hold if we replace AN by ETE. We show this below by presenting a simple strategyproof mechanism (d^1, τ^1) to be applied in a setting where there is one indivisible object and two agents $\{1, 2\}$. Recall that, by Fact 1, such a strategyproof mechanism is determined by the $K_i(.)$ and $T_i(.)$ functions associated with it. Suppose that for all i and all v,

$$K_1(v_2) = v_2 \text{ and } K_2(v_1) = \max\{0, v_1 - \epsilon\}$$

and

$$T_1(v_2) = v_2 + \epsilon \text{ and } T_2(v_1) = \max\{0, v_1 - \epsilon\}, \epsilon > 0$$

By Proposition 1, (d^1, τ^1) does not satisfy AN as $T_i(.)$ functions depend on the identity i. Further, it is easy to check that this mechanism satisfies AS and NBD. Also, note that for any $x \geq 0$, at any profile $v = (x, x)$, $d_1(v) = 0$, which implies that $u((d_1(v), \tau_1(v)); x) = K_1(x) = x$. Note that, irrespective of whether $d_2(v) = 1$ or not, then $u((d_2(v), \tau_2(v)); x) =$

x. Hence, (d^1, τ^1) is a mechanism, which does not employ a VARP allocation rule, and satisfies AS, ETE, NBD, SP - but not AN. Thus, our characterization of VARP allocation rule crucially depends on stronger implications of AN.

6 Conclusion

This paper provides a justification to reserve pricing at auctions using normative and strategic axioms unrelated to revenue considerations. In particular, it provides a topological interpretation of a reserve price as the *infimum* of the set of non-negative real numbers satisfying the following property: if all agents bid the same number from this set, then at least one object is allocated. Finally, it provides complete characterizations of VARP in single and multiple objects settings. Whether these results continue to hold in a multiple heterogeneous objects setting would be an interesting question for future research.

7 Appendix

7.1 Preliminary Results for $m \geq 1$ objects

Recall that for all $v \in \mathbb{R}^N_+$, $W(v)$ is the set of agents who get an object at the profile v. Since there are m objects to be allocated and not all objects are allocated at all profiles, $|W(v)| \leq m$, $\forall\, v \in \mathbb{R}^N_+$. The following proposition establishes for any anonymous, agent sovereign and strategyproof mechanism; that the threshold functions discussed in Fact 1 must be independent of respective agent labels.

Proposition 1. Any mechanism (d, τ) that satisfies AN, AS and SP must satisfy the following properties

1. $T_i(z) = T(z)$ for all $z \in \mathbb{R}^{n-1}_+$ and all $i \in N$.
2. $K_i(z) = K(z)$ for all $z \in \mathbb{R}^{n-1}_+$ and all $i \in N$.

Proof: Fix any mechanism (d, τ) that satisfies AN and SP. It is well known (see footnote 11 of Ashlagi and Serizawa [1]) that any mechanism (d, τ) satisfying the notion of AN defined in Definition 4 is equivalent to the requirement that: for all $i \neq j \in N$ and any two profiles v, v' such that $v_i = v'_j, v_j = v'_i, v_{-i-j} = v'_{-i-j}$;

$$u((d_i(v), \tau_i(v)); v_i) = u((d_j(v'), \tau_j(v')); v'_j).$$

We use this result to establish our proof.[23]

[23] This can be proven from Definition 4 by applying a simple bijection $\pi^{ij} : N \mapsto N$ on any profile v, where $\pi^{ij}(i) = j, \pi^{ij}(j) = i$ and for all $k \neq i, j$, $\pi^{ij}(k) = k$.

Suppose there exists some $z \in \mathbb{R}_+^{n-1}$ such that $T_1(z) \neq T_2(z)$. W. l. o. g. suppose that $T_1(z) > T_2(z)$. Construct the profile v such that $v_{-1} = z$ and $v_1 \in (T_2(z), T_1(z))$. Then, from Fact 1 it follows that $d_1(v) = 0$ since $v_1 < T_1(v_{-1}) = T_1(z)$ by construction. Now, consider the profile $v' = (v'_1, v'_2, v_{-1-2})$ where $v'_1 = v_2$ and $v'_2 = v_1$. Note that $v'_{-2} = v_{-1} = z$. Therefore, $d_2(v') = 1$ since $v'_2 = v_1 > T_2(z) = T_2(v'_{-2})$. Further, AN requires that $u(d_1(v), \tau_1(v); v_1) = u(d_2(v'), \tau_2(v'); v'_2)$. Then, from Fact 1 it follows that $v_1 + K_2(z) - T_2(z) = K_1(z)$ for all $v_1 \in (T_2(z), T_1(z))$. This leads to contradiction as for the supposed value of z, only one possible value of v_1 can satisfy this equality. Thus, arguing in this manner we can show that $T_i(z) = T_{i+1}(z)$ for all $i = 1, \ldots, n-1$ and all $z \in \mathbb{R}_+^{n-1}$. Therefore, statement (1) follows.
To prove statement 2, we fix any $z' \in \mathbb{R}_+^{n-1}$ and show that $K_1(z') = K_2(z')$. Consider the profile v such that $v_1 = v_2$ and $v_{-1} = v_{-2} = z'$. If $d_1(v) = d_2(v) = 0$ then by AN, $u(d_1(v), \tau_1(v); v_1) = u(d_2(v), \tau_2(v); v_2)$ implying $K_1(z') = K_2(z')$. If $d_1(v) = 1$ and $d_2(v) = 0$ then AN, Remark 1 and statement (1), imply that $v_1 + K_1(z') - T(z') = K_2(z')$ and $T(z') = v_1$ leading to the conclusion $K_1(z') = K_2(z')$. Similarly, if $d_1(v) = 0$ and $d_2(v) = 1$,we can show that $K_1(z') = K_2(z')$. Finally, if $d_1(v) = d_2(v) = 1$, then by AN and statement (1), $v_1 + K_1(z') - T(z') = v_2 + K_2(z') - T(z')$ implying $K_1(z') = K_2(z')$. Thus, arguing in this manner we can show that $K_i(z') = K_{i+1}(z')$ for all $i = 1, \ldots, n-1$ and all $z' \in \mathbb{R}_+^{n-1}$. Therefore, statement (2) follows. □

Proposition 2. For any mechanism (d, τ) that satisfies AN, AS and SP, the $K(.)$ and the $T(.)$ functions must be symmetric.

Proof: Fix any mechanism (d, τ) that satisfies AN, AS and SP. Suppose that there exists a $z \in \mathbb{R}_+^{n-1}$, an $i \in N$ and a bijection $\pi : N \setminus \{i\} \mapsto N \setminus \{i\}$ such that $T(z) \neq T(\pi z)$. W. l. o. g. assume that $T(z) < T(\pi z)$ and fix any $x \in (T(z), T(\pi z))$. Consider a bijection $\pi' : N \mapsto N$ such that $\pi' i = i$ and $\pi' j = \pi j$ for all $j \neq i$ and the profile v such that $v_i = x$ and $v_{-i} = z$. By Fact 1, $d_i(v) = 1$ and $d_i(\pi' v) = 0$ because $(\pi' v)_{-i} = \pi z$. By AN, $u(d_i(v), \tau_i(v); x) = u(d_i(\pi' v), \tau_i(\pi' v); x)$ implying that $x + K(z) - T(z) = K(\pi z)$. Since x was chosen arbitrarily from the interval $(T(z), T(\pi z))$, we get a contradiction.
To show the $K(.)$ function to be symmetric, consider any $i \in N$, any profile $v \in \mathbb{R}_+^n$ and any bijection $\hat{\pi} : N \mapsto N$ such that $\hat{\pi}(i) = i$. Since $T(.)$ function has already shown to be symmetric above, $T(v_{-i}) = T((\hat{\pi} v)_{-i})$ and so, either $v_i = T(v_{-i})$ or $d_i(v) = d_i(\hat{\pi} v)$. By AN, $u(d_i(v), \tau_i(v); v_i) = u(d_i(\hat{\pi} v), \tau_i(\hat{\pi} v); v_i)$, which implies that $K(v_{-i}) = K((\hat{\pi} v)_{-i})$. Hence, the result follows. □

The following lemma states that any mechanism satisfying AN, AS, NBD, and SP, must also satisfy a *weaker* form of efficiency which requires that whenever objects are allocated, the allocation must be efficient.

Lemma 1. A mechanism (d, τ) satisfies AN, AS, NBD, and SP only if $\forall\, v \in \mathbb{R}^n_+$, $i \neq j \in N$,

$$d_i(v) = 1 \text{ and } d_j(v) = 0 \implies v_i \geq v_j$$

Proof: Fix any mechanism satisfying AN, AS, NBD, and SP and any profile $v \in \mathbb{R}^n_+$. Suppose w. l. o. g. that $d_1(v) = 0$, $d_2(v) = 1$, $v_1 > v_2$. Consider the profile $\tilde{v} := (\tilde{v}_2, v_{-2})$ where $\tilde{v}_2 = v_1$. By SP, $d_2(v) = 1 \implies d_2(\tilde{v}) = 1$, and so, by NBD, $d_1(v) = 0 \implies d_1(\tilde{v}) = 0$. Therefore, by Remark 1, $T(v_{-2}) = v_1$. Arguing as before, by SP, $d_1(v_2, \tilde{v}_{-1}) = 0$ and so, by AN, $u(d_1(v_2, \tilde{v}_{-1}), \tau_1(v_2, \tilde{v}_{-1}); v_2) = u(d_2(v), \tau_2(v); v_2)$. Therefore, $K(v_{-2}) = v_2 + K(v_{-2}) - T(v_{-2}) \implies T(v_{-2}) = v_2 \neq v_1$ and hence, contradiction. Thus, the result follows. $\square$

Observation 1. From Lemma 1 it follows that for any mechanism satisfying AN, AS, NBD and SP, at any valuation profile $v \in \mathbb{R}^n_+$, if $d_i(v) = 0$ for some $i \in N$, then $d_j(v) = 0$ for all agents j such that $v_j < v_i$. Similarly, if there exists an $i \in N$ with $d_i(v) = 1$, then $d_j(v) = 1$ for all agents j such that $v_j > v_i$. Athanasiou [2] and Sprumont [27] show the same result in the single object setting without the use of NBD axiom. They accomplish this by exploiting a restriction implicit in single object setting, which implies that, at any profile, an agent getting the object implies that all other agents do not get any object. However, in multiple homogeneous objects setting, no such restriction is implicit. And so, we need the NBD axiom to tackle this additional complexity.[24]

The following lemma states the restriction imposed by NBD and SP axioms on the decision rule at profiles where all agents have bid the same value. It establishes the existence of a non-negative real number η such that no objects are allocated at any profile where (i) all agents have bid the same value and (ii) this value is less than η. Also, if all agents bid the same value that is greater than η, *at least* one object must be allocated.

[24]To see that Lemma 1 cannot hold without NBD, consider the following example. Suppose that $m = 2$, $N = \{1, 2, 3\}$, and consider a mechanism (d, τ) belonging to the class described by Fact 1 such that for all $i \in N$ and $z \in \mathbb{R}^{n-1}_+$,

$$K_i(z) = 0 \text{ and } T_i(z) = \begin{cases} \max\{z(2), 20\} & \text{if } z(1) < 10 \\ z(2) & \text{if } z(1) \geq 10 \end{cases}$$

Note that by Fact 1, (d, τ) satisfies SP and AS. Further, at any valuation profile, if valuations of any pair of agents are changed, their utilities get interchanged. Hence, (d, τ) satisfies AN. To see that this mechanism violates NBD, consider the valuation profile $(15, 10, 5)$. Note that according our decision rule $d(15, 10, 5) = (1, 1, 0)$. But if agent 2 unilaterally changes her reported valuation to 8, the decision changes to $d(15, 8, 5) = (0, 1, 0)$, and so, agent 2 continues to get the object but agent 1's decision outcome is affected as she no longer gets the object. Thus, (d, τ) does not satisfy NBD. Also, note that at profile $(15, 8, 5)$; the highest bidder does not get an object but the second highest bidder gets an object, which is a violation Lemma 1.

Lemma 2. A mechanism (d, τ) satisfies NBD and SP only if $\exists\, \eta \geq 0$ such that $\forall\, x \geq 0$,

$$x < \eta \implies \bar{x}^n \in B_0^m \text{ and } x > \eta \implies \bar{x}^n \notin B_0^m$$

Proof: Fix any mechanism (d, τ) satisfying NBD and SP. Suppose that there exist $0 \leq x < y$ such that $\bar{x}^n \notin B_0^m$ and $\bar{y}^n \in B_0^m$. W. l. o. g. suppose that $d_i(\bar{x}^n) = 1$ for all $i = 1, \ldots, l$ where $l \in \{1, \ldots, m\}$ (that is, l objects are allocated at profile $\bar{x}^n$). Define the sequence of profiles $(p^k)_{k=1}^l$ where $p^1 = (y, \bar{x}_{-1}^n)$ and for all $2 \leq k \leq l$, $p^k = (y, p_{-k}^{k-1})$. By NBD and SP, for all $1 \leq i \leq l$, $d_i(\bar{x}^n) = 1 \implies d_i(p^1) = 1 \implies d_i(p^2) = 1 \implies \ldots \implies d_i(p^l) = 1$ and so, $p^l \notin B_0^m$. Similarly construct another sequence of profiles $(q^k)_{k=l+1}^n$ such that $q^{l+1} = (x, \bar{y}_{-\{l+1\}}^n)$ and for all $l + 2 \leq k \leq n$, $q^k = (x, q_{-k}^{k-1})$. By SP and NBD, $y^n \in B_0^m \implies q^{l+1} \in B_0^m \implies q^{l+2} \in B_0^m \implies \ldots \implies q^n \in B_0^m$. By construction, $q^n = p^l$ and hence, contradiction. Therefore, for any $x \geq 0$, if $\bar{x}^n \notin B_0^m$ and then $\forall\, y > x$ it must be that $\bar{y}^n \notin B_0^m$. Thus, if the set $\{x \geq 0 : \bar{x}^n \notin B_0^m\}$ is non-empty, then the result follows from the choice of $\eta := \inf\{x \geq 0 : \bar{x}^n \notin B_0^m\}$. If $\{x \geq 0 : \bar{x}^n \notin B_0^m\} = \emptyset$ then no objects are allocated at any profile where all agents have bid the same value. In this case the result follows by assigning $\eta := \infty$. $\square$

The following lemma shows that if $\eta > 0$ then no object is allocated at any profile where the highest valuation is strictly less than η.

Lemma 3. A mechanism (d, τ) satisfies AN, NBD and SP only if $\forall\, v \in [0, \eta)^n$, $v \in B_0^m$.

Proof: Fix any mechanism (d, τ) satisfying AN, NBD and SP and any $v \in [0, \eta)^n$. W. l. o. g. assume that $v_1 \geq v_2 \geq \ldots \geq v_n$. By definition $v_1 < \eta$, and so, by Lemma 2, $\bar{v}_1^n \in B_0^m$. Construct a sequence of profiles $(p^k)_{k=1}^{n-1}$ such that $p^1 = (v_2, \bar{v}_{1_{-2}}^n)$ and for all $2 \leq k \leq n-1$, $p^k = (v_{k+1}, p_{-\{k+1\}}^{k-1})$. By SP and NBD, $\bar{v}_1^n \in B_0^m \implies p^1 \in B_0^m \implies \ldots \implies p^{n-1} \in B_0^m$. Note that by construction, $p^{n-1} = v$ and hence, the result follows. $\square$

7.2 Proof of Theorem 1

Fix any mechanism (d, τ) satisfies AN, AS, NBD and SP. By Lemma 2, there exists an $\eta := \inf\{x \geq 0 : \bar{x}^n \notin B_0^1\}$. Given Proposition 1 and Fact 1; the result would follow if we show that the threshold function $T(.)$ associated with (d, τ) is of the following form:

$$\textbf{(i)} \;\; T(v_{-i}) = \max\{v_{-i}(1), \eta\}, \forall\, i \in N, \forall\, v \in \mathbb{R}_+^N.$$

We establish the equality **(i)** in the following two steps.

Step 1

In this step we establish that $v_i < \max\{v_{-i}(1), \eta\} \implies d_i(v) = 0$, for all v and all i. In fact, this can easily be seen to follow from Lemma 2, Lemma 3, and Observation 1.

Step 2

In this step, we establish that $v_i > \max\{v_{-i}(1), \eta\} \implies d_i(v) = 1$, for all v and i.

To see this, fix any $i \in N$ and any profile $v \in \mathbb{R}^n_+$ such that $v_i > \max\{v_{-i}(1), \eta\}$. Note that, either $v_i = v(1) > v(2) > \eta$ or $v_i = v(1) > \eta \geq v(2)$. We analyze each of the two cases below, and show that in each case: $d_i(v) = 1$

`Case 1:` $v_i = v(1) > v(2) > \eta$
By Lemma 2, $\overline{v(2)}^n \notin B_0^1$ and so, from Remark 1 and Proposition 1, it follows that $T(\overline{v(2)}^{n-1}) = v(2)$. Construct a sequence of profiles $\{p^k\}_{k=1}^n$ such that $p^1 = \overline{v(2)}^n$, $p^2 = (v_i, p^1_{-i})$ and $\forall\, 3 \leq k \leq n$, $p^k = (v_{t_k}, p^{k-1}_{-t_k})$ where $t_k \in \{j \in N | v_j = v(k)\}$ (by the tie-breaking rule, this set is a singleton set). Further, $T(p^1_{-i}) = T(\overline{v(2)}^{n-1}) = v(2)$ and so under the supposition $v_i = v(1) > v(2)$, $d_i(p^2) = 1$. Since $m = 1$ it follows that $d_j(p^2) = 0, \forall\, j \neq i$. Moreover, by SP and NBD, for all $j \in N$, $d_j(p^2) = d_j(p^3) = \ldots = d_j(p^n)$. Since, by construction, $p^n = v$, we get that $d_i(v) = 1$ and $d_j(v) = 0$ for all $j \neq i$.

`Case 2:` $v_i = v(1) > \eta \geq v(2)$
Consider the sequence of profiles $\{p^k\}_{k=0}^n$ where $p^0 = \overline{\eta + \epsilon}^n$ and $\epsilon \in (0, v_i - \eta)$. For all $1 \leq k \leq n$, $p^k = (v_{t_k}, p^{k-1}_{-t_k})$ where $t_k \in \{j \in N | v_j = v(k)\}$ (as mentioned before, this set is a singleton set by the tie-breaking rule). Together with Remark 1, Lemma 2 and Proposition 1, we get that $p^0 \notin B_0^1$ which implies that $T(p^0_{-j}) = \eta + \epsilon$ for all $j \in N$. Further, by construction, $p^1_i = v_i$ and $p^0_{-i} = p^1_{-i}$. Therefore, $p^1_i > T(p^1_{-i}) = \eta + \epsilon$ and so, from Fact 1 it follows that $d_i(p^1) = 1$. Since $m = 1$, we can then claim that $d_j(p^1) = 0$ for all $j \neq i$. Hence, by SP and NBD, for all $j \in N$, $d_j(p^1) = d_j(p^2) = \ldots = d_j(p^n)$. By construction, $p^n = v$ which implies that $d_i(v) = 1$. □

7.3 Proof of Theorem 2

The sufficiency is easy to check. The necessity follows from Proposition 2 (in subsection 7.1 of Appendix) and Theorem 1. □

7.4 Proof of Theorem 3

Fix any continuous regular mechanism (d, τ) that satisfies AN, AS, NBD, SP. Given Proposition 1 and Fact 1; the result would follow if we show that the threshold function $T(.)$ associated with (d, τ) is of the following form:

$$(\mathbf{a})\;\; T(v_{-i}) = \max\{v_{-i}(m), \eta\}, \forall\, i \in N, \forall\, v \in \mathbb{R}^N_+,$$

where, as in proof of Theorem 1, $\eta := \inf\{x \geq 0 : \bar{x}^n \notin B_0^m\}$ and, as defined earlier, $B_0^m = \{v \in \mathbb{R}^N_+ : W^m(v) = \emptyset\}$. We establish $(\mathbf{a})$ in the following four steps:

Step 1

In this step we establish that the $T(.)$ function associated with mechanism (d, τ) is continuous.

Fix any sequence $\{z^k\}$ such that $z^k \in \mathbb{R}_+^{n-1}$ for all k, $\{z^k\} \to z^*$. Suppose that the sequence $\{T(z^k)\}$ does not converge to $T(z^*)$. Now, if there exists a z^k such that $T(z^k) < 0$, then for all $x > 0$, Lemma 1 implies that all agents be given objects at profile (x, z^k) leading to a contradiction as $m < n$. Also, if there exists a z^k such that $T(z^k) = \infty$, then for all values $x \geq 0$, $d_1(x, v_{-1}) = 0$ when $v_{-1} = z^k$ which violates AS. Therefore, we can infer that $\{T(z^k)\}$ is a bounded sequence.[25] And so, it must have a convergent subsequence. Therefore, *to simplify notation*, we can assume without loss of generality that $\{T(z^k)\}$ is a convergent sequence that has limit at some $\beta \geq 0$. By supposition, $\beta \neq T(z^*)$.[26]

If $\beta > T(z^*)$ then fix any $x \in (T(z^*), \beta)$. Note that there exists a subsequence $\{T(z^{k^l})\} \subseteq (\beta - \epsilon, \beta + \epsilon)$ for some particular $\epsilon \in (0, \beta - x)$, such that $\{T(z^{k^l})\} \to \beta$. Therefore, we can construct a sequence of profiles $\{v^l\}$ such that for all l, $v_1^l = x$ and $v_{-1}^l = z^{k^l}$. Now, since $\{z^k\}$ converges to z^*, the subsequence $\{z^{k^l}\}$ must also converge to z^*, and so, $\{v^l\}$ converges to (x, z^*). Therefore, for all l, $v_1^l = x < T(v_{-1}^l)$ which implies that $d_1(v^l) = 0$; but in limit $x > T(z^*)$ which implies that $d_1(x, z^*) = 1$. Further, by construction, $x \neq T(z^*)$ implying that agent is not indifferent between getting or not getting an object, at the limit profile (x, z^*). This contradicts the continuity of mechanism (d, τ). Arguing similarly, if $\beta < T(z^*)$, we arrive at a contradiction to continuity of (d, τ).

Step 2

In this step, we establish that by continuity of $T(.)$ functions, $\bar{\eta}^n \notin B_0^m$.

Note that by construction, for all $x > \eta$, $\bar{x}^n \notin B_0^m$, which implies that $x = T(\bar{x}^{n-1})$. Therefore, for any sequence $\{x^k\}$ such that $\{x^k\} \to \eta$ and $x^k > \eta, \forall\, k$; we get that $\{T(x^k, x^k, \ldots, x^k)\} \to \eta$. And so, by continuity of $T(.)$ function, we get that $T(\bar{\eta}^{n-1}) = \eta$. Hence, it follows from our tie-breaking rule that $\bar{\eta}^n \notin B_0^m$.

[25] In case the sequence $\{T(z^k)\}$ is unbounded, by AS, it would be unbounded above. Hence, there would exist a monotone increasing subsequence $\{T(z_l^k)\}_{l=1}^{\infty}$ such that it is *properly divergent*, or with some abuse of notation, $\{T(z_l^k)\} \to \infty$. Therefore, there exists an $M^* \in \mathbb{N}$ such that for all $l > M^*$, $T(z_l^k) > T(z^*) + 1$. Hence, for any sequence of profiles $\{v^t\}$ where for all t, $v_i^t = T(z^*) + 1$ and $v_{-i}^t = z_{M*+t}^k$, by Fact 1, $d_i(v^t) = 0$. Further, by construction, $\{v^t\}$ converges to v^* where $v_i^* = T(z^*) + 1$, and $v_{-i}^* = z^*$; and so, by Fact 1, $d_i(v^*) = 1$. Thus, by continuity, $u((1, \tau(v^*)); v_i^*) = u((0, \tau(v^*)); v_i^*)$ implying that $T(z^*) + 1 = T(z^*)$, which is a contradiction.

[26] Here our objective is to show that any arbitrary convergent subsequence of $\{T(z^k)\}$ must converge to $T(z^*)$; and then invoke Theorem 3.4.9 of Bartle and Sherbert [5] which states that any bounded sequence of real numbers, all of whose convergent subsequences converge to the same limit L, must also converge to L.

So, instead of introducing complicated new notations to denote a convergent subsequence of $\{T(z^k)\}$, we assume without loss of generality that the original sequence $\{T(z^k)\}$ converges to some real β, and then show below that $\beta = T(z^*)$.

Step 3

In this step we show that for any $z \in \mathbb{R}^{n-1}$ with $z(1) = \eta > z(n-1)$, $T(z) = \eta$.[27] Fix any such z. Since Proposition 2 has established $T(.)$ as a symmetric function, we can assume w.l.o.g. that $z_k = z(k)$ for all $k = 1, \ldots, n-1$. Fix the number $h \in \{1, \ldots, n-1\}$ such that $z_h \geq \eta > z_{h+1}$. For any $x \in [0, \eta]$, define

$$\tilde{z}^x := (\underbrace{\eta - x, \eta - x, \ldots, \eta - x}_{h \text{ coordinates}}, z_{h+1}, z_{h+2}, \ldots, z_n).$$

Now, consider the first possibility that $T(z) < \eta$. By continuity of $T(.)$ function, there exists an $\epsilon > 0$ such that for all $x \in (0, \epsilon)$, $T(\tilde{z}^x) < \eta$ and $\eta - x \geq z_{h+1}$. Now, if for all $x \in (0, \epsilon)$, $\eta - x \leq T(\tilde{z}^x)$, then we can construct a sequence $\{x^k\}$ such that for all k, $0 < x^k < \epsilon$ and $\{x^k\} \to 0$. Therefore, $\{\eta - x^k\} \to \eta$, and so, by continuity of $T(.)$, $\{T(\tilde{z}^{x^k})\} \to T(z)$ which implies that $T(z) \geq \eta$. However, by our supposition, $T(z) < \eta$ and so, we get a contradiction. Thus, there must exist an $x' \in (0, \epsilon)$ such that $\eta - x' > T(\tilde{z}^{x'})$. Therefore, the profile v' where $v'_1 = \eta - x'$ and $v'_{-1} = \tilde{z}^{x'}$ must have at least one object allocated to agent 1, implying that $v' \notin B_0^m$. However, $v' \in [0, \eta)^n$ and so, $v' \notin B_0^m$ contradicts Lemma 3. Therefore we can infer that $T(z) \geq \eta$.

Now, consider the possibility that $T(z) > \eta$. Then a profile v with $v_1 \in (\eta, T(z))$ and $v_{-1} = z$ will have $d_1(v) = 0$. By Lemma 1, we get that $d_i(v) = 0$ for all $i > 1$ implying that $v \in B_0^m$. By the regularity condition, no objects must be given out at this profile v when $m = 1$ which would contradict Theorem 1.

Step 4

In this step, we show that for all i and all v, $T_i(v_{-i}) = \max\{v_{-i}(m), \eta\}$.

Fix any $i \in N$ and any $v \in \mathbb{R}^n_+$. Consider the two possible cases $v_{-i}(m) \geq \eta$ and $v_{-i}(m) < \eta$. We accomplish the proof that $T(v_{-i}) = \max\{v_{-i}(m), \eta\}$ by showing that $T(v_{-i}) = v_{-i}(m)$ in the former case and $T(v_{-i}) = \eta$ in the latter case. For economy of notation, henceforth, we denote vector v_{-i} by z in the proof. By Proposition 2, $T(.)$ is symmetric and so, w.l.o.g. assume that $z_k = z(k)$ for all $k = 1, \ldots, n-1$. For simplicity of notation, define $\theta := z_m$.

Case 1: $\theta \geq \eta$
By Step 2 and construction, $\theta \geq \eta \implies \bar{\theta}^n \notin B_0^m$, and so, by our tie-breaking rule $W(\bar{\theta}^n) = \{1, 2, \ldots, m\}$. Consider the sequence of profiles $(q^k)_{k=0}^{n-2}$ such that $q^0 = \bar{\theta}^n$, for all $1 \leq k \leq m-1$, $q^k = (z_k, q^{k-1}_{-\{k\}})$ and for all $m \leq k \leq n-2$, $q^k = (z_{k+1}, q^{k-1}_{-\{k+2\}})$. From SP and NBD it follows that $\forall\, j \in N$, $d_j(q^0) = d_j(q^1) = \ldots = d_j(q^{n-2})$. Therefore, we get that $d_j(q^{n-2}) = 1$, $\forall\, j = 1, \ldots, m$ and $d_j(q^{n-2}) = 0$, $\forall\, j = m+1, \ldots, n$. Also,

[27] Recall that for any vector $x \in \mathbb{R}^k$ where $k \in \mathbb{N}$, $x(i)$ is the ith greatest coordinate in x. To make this notation well defined, w.l.o.g., we break ties using the order $1 \succ 2 \succ \ldots \succ n$.

by construction, $q^{n-2}_m = q^{n-2}_{m+1} = \theta$ and $q^{n-2}_{-\{m\}} = q^{n-2}_{-\{m+1\}} = z$. Thus, arguing as in Remark 1, the fact that $d_m(q^{n-2}) = 1, d_{m+1}(q^{n-2}) = 0$ can be used to infer that $\theta \geq T(q^{n-2}_{-\{m\}}) = T(q^{n-2}_{-\{m+1\}}) \geq \theta$ implying that $T(z) = \theta = z_m$.

Case 2: $\theta < \eta$
Fix any $x \geq 0$, define the profile p^x such that $p^x_1 = x$ and $p^x_{-1} = z$ and construct the profile p such that $p_k = p^x(k)$ for all $k \in \{1, 2, \ldots, n\}$. Therefore, by construction, $p_1 \geq p_2 \geq \ldots \geq p_n$. We consider two subcases: $x \geq \eta$ and $x < \eta$. In the following paragraphs, we show that $d_1(p^x) = 1$ in the former case while $d_1(p^x) = 0$ in the latter case. By Fact 1, this inference to imply that $T(p^x_{-1}) = T(z) = \eta$.

Subcase 1. $x \geq \eta$
Define the agent $g := \{j \in N : p_j \geq \eta \text{ and } p_{j+1} < \eta\}$. Since $\theta < \eta$ and $x > \eta$, agent g is well defined and $g \in \{1, \ldots, m\}$. Therefore, p_g is the smallest coordinate of p greater than or equal to η while p_{g+1} is the largest coordinate of p strictly less that η. Consider a sequence of profiles $\{u^k\}^n_{k=0}$ such that $u^0 = \bar{\eta}^n$, for all $1 \leq k \leq n-g$, $u^k = (p_{g+k}, u^{k-1}_{-(g+k)})$, and for all $n - g + 1 \leq k \leq n$, $u^k = (p_{n+1-k}, u^{k-1}_{-(n+1-k)})$. Note that by Step 3, for $k \in \{1, \ldots, n-g\}$, $T(u^k_{-i}) = \eta$ for all $i \in N$. So, if $i \in \{1, \ldots, g\}$ then $d_i(u^{n-g}) = 1$, or else $d_i(u^{n-g}) = 0$. Arguing as before, by NBD and SP, $d_i(u^{n-g}) = d_i(u^n)$ for all $i \in N$. Since by construction, $u^n = p$ and so we have established that

$$d_i(p) = \begin{cases} 1 & \forall\, i = 1, \ldots, g \\ 0 & \forall\, i = g+1, \ldots, n \end{cases} \tag{1}$$

Now, by supposition $x \geq \eta$ and so, $x \in \{p_1, \ldots, p_g\}$ which implies that the agent bidding x at profile p gets an object. Since, the $T(.)$ function is symmetric, we can infer that $d_1(p^x) = 1$.

Subcase 2. $x < \eta$
If $x < \eta$, there arise two cases: $p_1 < \eta$ and $p_1 \geq \eta$. If $p_1 < \eta$ then, by construction, $p^x \in [0, \eta)^n$ and so, from Lemma 3, $p^x \in B^m_0$ implying that $d_1(p^x) = 0$. If $\eta \leq p_1$ then agent g is well defined as in the subcase 1 above. And so, we can argue as above to obtain equation (1). Note that $x < \eta \implies x \in \{p_{g+1}, \ldots, p_n\}$ which implies that the agent bidding x at profile p does not get an object. Therefore, as before, from symmetry of $T(.)$ function it follows that $d_1(p^x) = 0$. □

7.5 Proof of Theorem 4

The proof of sufficiency is easy to check. The proof of necessity follows from Proposition 2 (in subsection 7.1 of Appendix) and Theorem 3. □

7.6 Relation between non bossiness in decision (NBD) and Satterthwaite and Sonnenschein [26] version of non-bossiness (SSNB).

Lemma 4. *If a strategyproof mechanism (d,t) violates NBD then it violates SSNB.*

Proof: Fix any mechanism (d,t) that violates NBD. Therefore, there exists $i \in N$, $v_{-i} \in \mathbb{R}^{N\setminus\{i\}}$, and $x' \neq y' \geq 0$ such that

$$d_i(x', v_{-i}) = d_i(y', v_{-i}) \text{ and } \exists\, j \in N \setminus \{i\} \text{ such that } d_j(x', v_{-i}) \neq d_j(y', v_{-i}).$$

Now it is well known that in a discrete object allocation problem with unit demand, a strategyproofness mechanism exhibits the property that $d_i(a, v_{-i}) = d_i(b, v_{-i}) \implies t_i(a, v_{-i}) = t_i(b, v_{-i})$ for all $a, b \geq 0$, all i, and all v_{-i} (because definition of strategyproofness requires that $a\{d_i(b, v_{-i}) - d_i(a, v_{-i})\} \leq \{\tau_i(a, v_{-i}) - \tau_i(b, v_{-i})\} \leq b\{(d_i(b, v_{-i}) - d_i(a, b_{-i})\}$ for all $a, b \geq 0$, all i, and all v_{-i}). Therefore, it must be that $t_i(x', v_{-i}) = t_i(y', v_{-i})$. But now we have a pair of agents i, j such that when i changes her reported valuation unilaterally from x' to y' (with all others reporting v_{-i}), her consumption bundle remains unchanged (as $(d_i(x', v_{-i}), t_i(x', v_{-i})) = (d_i(x', v_{-i}), t_i(x', v_{-i})))$ - but j' consumption bundle changes as her object assignments $d_j(x', v_{-i}) \neq d_j(y', v_{-i})$. Thus, we get a violation of SSNB. □

Hence, within the class of strategyproof mechanisms, NBD is weaker than SSNB.

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